RATIONAL MAPS WHOSE JULIA SETS ARE CANTOR CIRCLES

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Abstract. In this paper, we give a family of rational maps whose Julia sets are Cantor circles and show that every rational map whose Julia set is a Cantor set of circles must be topologically conjugate to one map in this family on their corresponding Julia sets. In particular, we give the specific expressions of some rational maps whose Julia sets are Cantor circles, but they are not topologically conjugate to any McMullen maps on their Julia sets. Moreover, some non-hyperbolic rational maps whose Julia sets are Cantor circles are also constructed.

1. Introduction

The study on the topological properties of the Julia sets of rational maps is a central problem in complex dynamics. For each degree at least two polynomial with disconnected Julia set, it was proved that all but countably many components of the Julia set are single points in [QY]. For the rational maps, the Julia sets may exhibit more complex topology. Pilgrim and Tan proved that if the Julia set of a hyperbolic (more generally, geometrically finite) rational map is disconnected, then, with the possible exception of finitely many periodic components and their countable collection of preimages, every Julia component is either a point or a Jordan curve [PT, Theorem 1.2]. In this paper, we will consider one class of rational maps whose Julia sets possess simple topological structure: each Julia component is a Jordan curve.

A subset of the Riemann sphere $\overline{\mathbb{C}}$ is called a *Cantor set of circles* (sometimes *Cantor circles* in short) if it consists of uncountably many closed Jordan curves which is homeomorphic to $\mathcal{C}\times\mathbb{S}^1$, where $\mathcal{C}$ is the middle third Cantor set and $\mathbb{S}^1$ is the unit circle. The first example of rational map whose Julia set is a Cantor set of circles was discovered by McMullen (See [Mc, §7]). He showed that if $f(z)=z^2+\lambda/z^3$ and λ is small enough, then the Julia set of f is a Cantor set of circles. Later, many authors focus on the following family, which is commonly referred as the *McMullen maps*:

$$g_\eta(z)=z^k+\eta/z^l, \tag{1.1}$$

where $k,l\geq 2$ and $\eta\in\mathbb{C}\setminus\{0\}$ (See [DLU, St, QWY] and the references there in). These special rational maps can be viewed as a perturbation of the simple polynomial $g_0(z)=z^k$ if η is small. It is known that when $1/k+1/l<1$, there exists a punched neighborhood $\mathcal{M}$ centered at origin in the parameter space, which is called the *McMullen domain*, such that when $\eta\in\mathcal{M}$, then the Julia set of g_η is a Cantor set of circles (See [Mc, §7] for $k=2,l=3$ and [DLU, §3] for the general cases).

There are following three natural questions: (1) Besides McMullen maps, does there exist any other rational maps whose Julia sets are Cantor circles? (2) If the answer of first question is yes, what do they like? Or in other words, can we find out their specific expressions? (3) Can we find out all the rational maps whose Julia sets are Cantor circles in some sense? This paper will give affirmative answers to these questions.

By quasiconformal surgery, we can obtain many new rational maps after disturbing the immediate super-attracting basin centered at ∞ of g_η into a geometric one. Fix one of them, then this map is not topologically conjugate to g_η on the whole $\overline{\mathbb{C}}$. But they are topologically conjugate to each other on their corresponding Julia sets. In particular, $h_{c,\eta}(z)=\frac{1}{z}\circ(z^k+c)\circ\frac{1}{z}+\eta/z^l$

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is an example, where $1/k + 1/l < 1$ and $c, \eta \in \mathbb{C} \setminus \{0\}$ are both small enough. However, these types of rational maps can be also regarded as the McMullen maps essentially, which are not we want to find since they can be obtained by doing a surgery only on the Fatou sets of the genuine McMullen maps. So it will be very interesting to find out other types of rational maps with Cantor circles Julia sets which are not topologically conjugate to any McMullen maps on their corresponding Julia sets.

The existence of "essentially" different types of rational maps from McMullen maps was known (See [HP, §§1,2]). In this paper, we will give the specific expressions of these types of rational maps, not only including the cases discussed in [HP], but also covering all the rational maps whose Julia sets are Cantor circles "essentially" (See Theorem 1.2). Let $p \in \{0, 1\}$, $n \geq 2$ be an integer and $d_1, \cdots, d_n$ be n positive integers such that $\sum_{i=1}^n \frac{1}{d_i} < 1$. We define

$$f_{p,d_1,\cdots,d_n}(z) = z^{(-1)^{n-p} d_1} \prod_{i=1}^{n-1} (z^{d_i + d_{i+1}} - a_i^{d_i + d_{i+1}})^{(-1)^{n-i-p}}, \tag{1.2}$$

where $a_1, \cdots, a_{n-1}$ are $n-1$ small complex numbers satisfying $0 < |a_1| < \cdots < |a_{n-1}| < 1$. In particular, if $n = 2$, then $f_{1,d_1,d_2}(z) = z^{d_2} - a_1^{d_1+d_2}/z^{d_1}$ is the McMullen map which is well studied by many authors. Moreover, $f_{0,d_1,d_2}(z) = z^{d_1}/(z^{d_1+d_2} - a_1^{d_1+d_2})$ is conformally conjugated to the McMullen map $z \mapsto z^{d_1} + \eta/z^{d_2}$ for some $\eta \neq 0$. The degrees of $f_{p,d_1,\cdots,d_n}$ at 0 and ∞ are d_1 and d_n respectively and $\deg(f_{p,d_1,\cdots,d_n}) = \sum_{i=1}^n d_i$. For each element in the family (1.2), it is easy to check that 0 and ∞ belong to the Fatou set of $f_{p,d_1,\cdots,d_n}$. Let D_0 and D_∞ be the Fatou components containing 0 and ∞ respectively. There are four cases:

(1) If $p = 1$ and n is odd, then $f(D_0) = D_0$ and $f(D_\infty) = D_\infty$;
(2) If $p = 1$ and n is even, then $f(D_0) = D_\infty$ and $f(D_\infty) = D_\infty$;
(3) If $p = 0$ and n is odd, then $f(D_0) = D_\infty$ and $f(D_\infty) = D_0$;
(4) If $p = 0$ and n is even, then $f(D_0) = D_0$ and $f(D_\infty) = D_0$.

Firstly we will find out suitable parameters a_i in (1.2), where $1 \leq i \leq n-1$, such the Julia set of each $f_{p,d_1,\cdots,d_n}$ in the four cases stated above is a Cantor set of circles. Let $\xi = \sum_{i=1}^n \frac{1}{d_i}$ and $K \geq 3$ be the maximal number of $d_1, \cdots, d_n$.

Theorem 1.1. *If $|a_{n-1}| = (s_1 K^{-2})^{1/d_n}$ and $|a_i| = (s_1 K^{-5})^{1/d_{i+1}} |a_{i+1}|$ for $1 \leq i \leq n-2$, where $s_1 > 0$ is small enough, then the Julia set of $f_{1,d_1,\cdots,d_n}$ is a Cantor set of circles. If $|a_{n-1}| = (s_0^{1/d_n + (1-\xi)/3})^{1/d_n}$ and $|a_i| = (s_0^{1+1/d_n+2(1-\xi)/3})^{1/d_{i+1}} |a_{i+1}|$ for $1 \leq i \leq n-2$, where $s_0 > 0$ is small enough, then the Julia set of $f_{0,d_1,\cdots,d_n}$ is a Cantor set of circles.*

To some extent, Theorem 1.1 means that we have found a family of rational maps whose Julia sets are Cantor circles with parameters s_1 and s_0. These rational maps can be seen as the perturbations of z^{d_n} or z^{-d_n} (according to $p = 1$ or 0) since s_1 and s_0 can be arbitrarily small. The specific value ranges of s_1 and s_0 are given in Section 2 (See (2.1) and (2.2)). Moreover, it will be shown that if $n \geq 3$, then each $f_{p,d_1,\cdots,d_n}$ is not topologically conjugate to any McMullen maps on their corresponding Julia sets (See Theorem 2.5). This means that we have found the specific expressions of rational maps whose Julia sets are Cantor circles which is "essentially" different from McMullen maps.

Note that if the Julia set $J(f)$ of a rational map f is a Cantor set of circles, then there exists no critical points on the $J(f)$ since each Julia component is a Jordan closed curve (See Lemma 3.1). This means that every periodic Fatou component of f must be attracting or parabolic. In fact, we have

Theorem 1.2. *Let f be a rational map whose Julia set is a Cantor set of circles. Then there exist $p \in \{0, 1\}$, positive integers $n \geq 2$ and $d_1, \cdots, d_n$ satisfying $\sum_{i=1}^n \frac{1}{d_i} < 1$ such that f is topologically conjugate to $f_{p,d_1,\cdots,d_n}$ on their corresponding Julia sets for suitable parameters a_i appeared in Theorem 1.1, where $1 \leq i \leq n-1$.*

Since the dynamics on the Fatou set can be disturbed freely, it follows from Theorem 1.2 that we have found "all" the possible rational maps whose Julia sets are Cantor circles. A rational map is *hyperbolic* if all critical points are attracted by attracting periodic orbits. For the regularity of the Julia components of $f_{p,d_1,\cdots,d_n}$, it can be shown that each Julia component of $f_{p,d_1,\cdots,d_n}$ is a quasicircle if $f_{p,d_1,\cdots,d_n}$ is hyperbolic (See Corollary 3.3).

From the topological point of view, all Cantor sets of circles are the same since they are all topologically equivalent to the "stand" Cantor set of circles $\mathcal{C}\times\mathbb{S}^1$, where $\mathcal{C}$ is the middle third Cantor set and $\mathbb{S}^1$ is the unit circle. Therefore, to obtain much richer structure of all Cantor sets of circles, we can look at the Cantor circles equipped with metric from the point of view of quasiconformal geometry. In fact, a basic problem in the quasiconformal geometry is to determine whether two given homeomorphic metric spaces are quasisymmetrically equivalent to each other.

Let (X,d_X) and (Y,d_Y) be two metric spaces. If there exist two homeomorphisms $f:X\to Y$ and $\zeta:[0,\infty)\to[0,\infty)$ such that $d_Y(f(x),f(y))/d_Y(f(x),f(z))\leq\zeta(d_X(x,y)/d_X(x,z))$ for every distinct points $x,y,z\in X$. Then (X,d_X) and (Y,d_Y) are called *quasisymmetrically equivalent* to each other. The *conformal dimension* $\mathrm{confdim}(X)$ of X is the infimum of the Hausdorff dimensions of all metric spaces which are quasisymmetrically equivalent to X.

Let $J_{p,d_1,\cdots,d_n}$ be the Julia set of $f_{p,d_1,\cdots,d_n}$ for $n\geq 2$. In the following, we always assume that a_i is chosen like in Theorem 1.1 such that $J_{p,d_1,\cdots,d_n}$ is a Cantor set of circles since we are only interested in this case. Meantime, we assume that η is small enough such the Julia set $J_{k,l}$ of McMullen map g_η defined as in (1.1) is a Cantor set of circles, where $1/k+1/l<1$. If $d_i=n+1$ for every $1\leq i\leq n$, we use f_n to denote $f_{p,n+1,\cdots,n+1}$ and let J_n be its corresponding Julia set.

Theorem 1.3. *The conformal dimension of $J_{p,d_1,\cdots,d_n}$ is $\mathrm{confdim}(J_{p,d_1,\cdots,d_n})=1+\alpha_{p,d_1,\cdots,d_n}$, where $\alpha_{p,d_1,\cdots,d_n}$ is the unique positive root of*

$$\sum_{i=1}^{n} d_i^{-\alpha_{p,d_1,\cdots,d_n}}=1. \tag{1.3}$$

In particular, if $d_i=n+1$ for every $1\leq i\leq n$, then $\alpha_n:=\alpha_{p,d_1,\cdots,d_n}=\log(n)/\log(n+1)$. If $m\neq n$, then $\alpha_m\neq\alpha_n$. If $n\geq 3$, then $\alpha_n\neq\alpha_{k,l}$ for every $k,l\geq 2$ such that $1/k+1/l<1$.

Theorem 1.3 gives a specific example to verify that there exist hyperbolic rational maps whose Julia sets are Cantor circles and whose conformal dimensions are arbitrarily close to 2 (See [HP, Theorem 2]). From the proof of Theorem 1.1, we know that all $f_{p,d_1,\cdots,d_n}$ are hyperbolic. Note that the conformal dimension is an invariant of the quasisymmetric class of a metric space (See [MT]) and the Julia set of every hyperbolic rational map has Hausdorff dimension strictly less than 2 (See [Su, Theorem 4 and Corollary]). Therefore, Theorem 1.3 has following two immediate corollaries:

Corollary 1.4. *For any $m,n\geq 2$, the Julia sets J_m and J_n are quasisymmetrically equivalent to each other if and only if $m=n$. Moreover, if $n\geq 3$, then J_n is not quasisymmetrically equivalent to any $J_{k,l}$ for $1/k+1/l<1$.*

Corollary 1.5. *The Hausdorff dimension $\mathrm{Hdim}(J_n)$ of J_n satisfies*

$$1+\log(n)/\log(n+1)\leq\mathrm{Hdim}(J_n)<2.$$

If η is small enough, then g_η is hyperbolic (See [DLU]). Now we construct some non-hyperbolic rational maps whose Julia sets are Cantor circles. Let $m,n\geq 2$ be two positive integers satisfying $1/m+1/n<1$ and $\lambda\in\mathbb{C}\setminus\{0\}$, we define

$$P_\lambda(z)=\frac{\frac{1}{n}((1+z)^n-1)+\lambda^{m+n}z^{m+n}}{1-\lambda^{m+n}z^{m+n}}. \tag{1.4}$$

It is straightforward to verify that 0 is a parabolic fixed point of P_λ with multiplier 1. We have

Theorem 1.6. *If $0<|\lambda|\leq 1/(2^{10m}n^3)$, then P_λ is non-hyperbolic and its Julia set is a Cantor set of circles.*

Inspired by Theorem 1.1, we can construct more non-hyperbolic rational maps such the Julia sets of them are Cantor circles. For simplicity, for each $n \geq 2$, we only consider the case $d_i = n+1$ for every $1 \leq i \leq n$. For every $n \geq 2$, we define

$$P_n(z) = A_n \, \frac{(n+1)z^{(-1)^{n+1}(n+1)}}{nz^{n+1}+1} \prod_{i=1}^{n-1} (z^{2n+2} - b_i^{2n+2})^{(-1)^{i-1}} + B_n, \tag{1.5}$$

where $b_1, \cdots, b_{n-1}$ are $n-1$ small complex numbers satisfying $1 > |b_1| > \cdots > |b_{n-1}| > 0$ and

$$A_n = \frac{1}{1+(2n+2)C_n} \prod_{i=1}^{n-1} (1 - b_i^{2n+2})^{(-1)^i}, \; B_n = \frac{(2n+2)C_n}{1+(2n+2)C_n} \text{ and } C_n = \sum_{i=1}^{n-1} \frac{(-1)^{i-1} b_i^{2n+2}}{1 - b_i^{2n+2}}. \tag{1.6}$$

The terms A_n and B_n here can guarantee that $P_n(1) = 1$ and $P_n'(1) = 1$. Namely, 1 is a parabolic fixed point of P_n with multiplier 1 (See Lemma 6.1).

Theorem 1.7. *For every $n \geq 2$ and $1 \leq i \leq n-1$, if $|b_i| = s^i$ for $0 < s \leq 1/(25n^2)$, then P_n is non-hyperbolic and its Julia set is a Cantor set of circles.*

It can be seen later the dynamics of P_n on their Julia sets are conjugated to that of f_n for every $n \geq 2$ ($p = 1$). One of the difference between their dynamics on the Fatou sets is the super-attracting basin of f_n at ∞ is replaced by a parabolic basin of P_n.

This paper is organized as follows: In section 2, we do some estimates and prove Theorem 1.1. In section 3, we prove Theorem 1.2. In section 4, we consider the quasisymmetric geometric of Cantor circles and prove Theorem 1.3 and Corollaries 1.4 and 1.5. In section 5, we show that the Julia set of P_λ is a Cantor set of circles if λ is small enough and prove Theorem 1.6. We will prove Theorem 1.7 in section 6 and leave a key lemma to the last section.

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Notations. We will use the following notations throughout the paper. Let $\mathbb{C}$ be the complex plane and $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the Riemann sphere. For $r > 0$ and $a \in \mathbb{C}$, let $\mathbb{D}(a,r) := \{z \in \mathbb{C} : |z-a| < r\}$ be the Euclidean disk centered at a with radius r. In particular, let $\mathbb{D}_r := \mathbb{D}(0,r)$ be the disk centered at the origin with radius r and $\mathbb{T}_r := \partial\mathbb{D}_r$ be the boundary of $\mathbb{D}_r$. As usual, $\mathbb{D} := \mathbb{D}_1$ and $\mathbb{S}^1 := \mathbb{T}_1$ denote the unit disk and the unit circle, respectively. For $0 < r < R < +\infty$, let $\mathbb{A}_{r,R} := \{z \in \mathbb{C} : r < |z| < R\}$ be the round annulus centered at the origin.

2. Location of the critical points and the hyperbolic case

First we give some basic and useful estimations.

Lemma 2.1. *Let $n \geq 2$ be an integer, $a \in \mathbb{C} \setminus \{0\}$ and $0 < \varepsilon < 1/2$.*
(1) *If $|z-a| \leq \varepsilon|a|$, then $|z^n - a^n| \leq ((1+\varepsilon)^n - 1)\,|a|^n$;*
(2) *If $|z^n - a^n| \leq \varepsilon|a|^n$, then $|a/z|^n < 1 + 2\varepsilon$ and $|z - ae^{2\pi i j/n}| < \varepsilon|a|$ for some $1 \leq j \leq n$;*
(3) *If $0 < \varepsilon < 1/n$, then $n\varepsilon < (1+\varepsilon)^n - 1 < 3n\varepsilon$ and $n\varepsilon/3 < 1 - (1-\varepsilon)^n < n\varepsilon$.*

Proof. Let $z = a(1 + re^{i\theta})$ for $0 \leq r \leq \varepsilon$ and $0 \leq \theta < 2\pi$, then

$$|z^n - a^n| = |(1+re^{i\theta})^n - 1| \cdot |a|^n \leq ((1+\varepsilon)^n - 1)\,|a|^n.$$

This proves (1). The first statement in (2) follows from $|a/z|^n \leq 1/(1-\varepsilon) < 1 + 2\varepsilon$ if $0 < \varepsilon < 1/2$. For the second statement, let $z^n = a^n(1+re^{i\theta})$ for $0 \leq r \leq \varepsilon$ and $0 \leq \theta < 2\pi$, then $z = ae^{2\pi i j/n}(1+re^{i\theta})^{1/n}$ for some $1 \leq j \leq n$ and we have

$$|z - ae^{2\pi i j/n}| = |(1+re^{i\theta})^{1/n} - 1| \cdot |a| \leq ((1+\varepsilon)^{1/n} - 1) \cdot |a| < \varepsilon|a|$$

if $n \geq 2$. The claim (3) can be proved by using Lagrange's mean value theorem to $x \mapsto x^n$ on the intervals $[1, 1+\varepsilon]$ and $[1-\varepsilon, 1]$ respectively. The proof is completed. $\square$

Fix $n \geq 2$ and let $d_1, \cdots, d_n \geq 2$ be n positive numbers such that $\xi = \sum_{i=1}^n \frac{1}{d_i} < 1$. Recall that $K \geq 3$ is the maximal number among $d_1, \cdots, d_n$. Let $u_1 = s_1 K^{-5}$ and $v_1 = s_1 K^{-2}$, where

$$0 < s_1 \leq \min\{K^{-5\xi/(1-\xi)}, K^{5-2K}\} < 1. \tag{2.1}$$

Let $u_0 = s_0^{1+1/d_n+2(1-\xi)/3}$, $v_0 = s_0^{1/d_n+(1-\xi)/3}$, where

$$0 < s_0 \leq \min\{2^{-(1-\xi)^{-1}(1+1/d_n-2\xi/3)^{-1}}, (4K)^{-3/(1-\xi)}, K^{-2K(1+1/d_n+2(1-\xi)/3)^{-1}}\} < 1. \tag{2.2}$$

For $p \in \{0, 1\}$, let $|a_{n-1,p}| = v_p^{1/d_n}$ and $|a_{i,p}| = u_p^{1/d_{i+1}} |a_{i+1,p}|$ be the $n-1$ parameters in the family $f_{p,d_1,\cdots,d_n}$, where $1 \leq i \leq n-2$. Since the cases $p = 0$ and $p = 1$ can be discussed uniformly in generally, we use s, u, v and a_i, respectively, to denote s_p, u_p, v_p and $a_{i,p}$ for simplicity when the situation is clear, where $1 \leq i \leq n-1$.

Lemma 2.2. (1) $u^{2/K} \leq K^{-4}$.

(2) *If* $1 \leq j \leq i \leq n-1$, *then* $|a_j/a_i| \leq u^{\frac{i-j}{K}}$.

(3) *If* $p = 1$, *then*

(3a) $(s/|a_1|)^{d_1} < su/(2v) = sK^{-3}/2$ *and*

(3b) $(|a_1|/s)^{d_1} v/2 > K$.

(4) *If* $p = 0$, *then*

(4a) $2Ku/v < s$ *and* $1/(2Kv) > (2/s)^{1/d_n}$;

(4b) $(s/|a_1|)^{d_1} < sv/2 < u^{1/2}/2$ *and*

(4c) $(|a_1|/s)^{d_1} u/(2v) > (2/s)^{1/d_n}$.

Proof. (1) From (2.1) and (2.2), we have $s_1 \leq K^{5-2K}$ and $s_0 \leq K^{-2K(1+1/d_n+2(1-\xi)/3)^{-1}}$. This means that $u_1^{2/K} = (s_1 K^{-5})^{2/K} \leq K^{-4}$ and $u_0^{2/K} \leq K^{-4}$.

(2) If $j = i$, then (2) is trivial. Suppose that $1 \leq j < i \leq n-1$, then

$$|a_j/a_i| = u^{\frac{1}{d_{j+1}} + \cdots + \frac{1}{d_i}} \leq u^{\frac{i-j}{K}}$$

since $K \geq d_i$ for $1 \leq i \leq n$. This proves (2).

(3) If $p = 1$, then $u = sK^{-5}$ and $v = sK^{-2}$. Since $s \leq K^{-5\xi/(1-\xi)}$, we have $s^{1-\xi} K^{5\xi} \leq 1$, so

$$s^{1-\frac{1}{d_1}} s^{-(\frac{1}{d_2} + \cdots + \frac{1}{d_n})} K^{5(\frac{1}{d_2} + \cdots + \frac{1}{d_{n-1}}) + \frac{2}{d_n}} 2^{\frac{1}{d_1}} K^{\frac{3}{d_1}} < 1.$$

This is equivalent to $s^{1-\frac{1}{d_1}} 2^{\frac{1}{d_1}} K^{\frac{3}{d_1}} / |a_1| < 1$ since

$$|a_1| = u^{\frac{1}{d_2} + \cdots + \frac{1}{d_{n-1}}} v^{\frac{1}{d_n}} = s^{\frac{1}{d_2} + \cdots + \frac{1}{d_n}} / K^{5(\frac{1}{d_2} + \cdots + \frac{1}{d_{n-1}}) + \frac{2}{d_n}}.$$

So we have $(s/|a_1|)^{d_1} < su/(2v) = sK^{-3}/2$ and (3a) is proved. Moreover, (3b) can be derived from (3a) directly since $(|a_1|/s)^{d_1} > 2K^3/s = 2K/v$.

(4) If $p = 0$, then $u = s^{1+1/d_n+2(1-\xi)/3}$, $v = s^{1/d_n+(1-\xi)/3}$. From (2.2), we know $4Ks^{(1-\xi)/3} \leq 1$, which means $2Ku/v = 2Ks^{1+(1-\xi)/3} < s$. Note that $2^{1+1/d_n} Ks^{(1-\xi)/3} < 1$, which is equivalent to $1/(2Kv) > (2/s)^{1/d_n}$. This ends the proof of (4a).

From (2.2), we know that

$$\begin{aligned}
1 &\geq 2s^{(1-\xi)(1+1/d_n-2\xi/3)} > 2^{\frac{1}{d_1}} s^{(1-\xi)(1+1/d_n-2\xi/3)} \\
&= 2^{\frac{1}{d_1}} s^{1-\frac{1}{d_1}} / s^{(\frac{1}{d_2} + \cdots + \frac{1}{d_{n-1}}) + \frac{1}{d_n}(\frac{1}{d_1} + \cdots + \frac{1}{d_n}) + \frac{2\xi(1-\xi)}{3}} \\
&> 2^{\frac{1}{d_1}} s^{1-\frac{1}{d_1}} / s^{(\frac{1}{d_2} + \cdots + \frac{1}{d_{n-1}}) + \frac{1}{d_n}(\frac{1}{d_1} + \cdots + \frac{1}{d_n}) + \frac{1-\xi}{3}(\frac{1}{d_1} + 2(\frac{1}{d_2} + \cdots + \frac{1}{d_{n-1}}) + \frac{1}{d_n})} \\
&= s^{1-\frac{1}{d_1}} (2/v)^{\frac{1}{d_1}} / |a_1|.
\end{aligned}$$

This means that $(s/|a_1|)^{d_1} < sv/2 = u^{1/2} s^{(1+1/d_n)/2}/2 < u^{1/2}/2$. So (4b) holds.

The proof of (4c) is similar to (4b). We just need to note that

$$1 \geq 2s^{(1-\xi)(1+1/d_n-2\xi/3)} > 2^{\frac{1}{d_1}(1+\frac{1}{d_n})} s^{(1-\xi)(1+1/d_n-2\xi/3)} > (s/|a_1|)(2v/u)^{\frac{1}{d_1}} (2/s)^{\frac{1}{d_1 d_n}}.$$

This means that $(|a_1|/s)^{d_1} u/(2v) > (2/s)^{1/d_n}$. □

In the following, we use f to denote $f_{p,d_1,\cdots,d_n}$ for simplicity. Note that 0 and ∞ are critical points of f with multiplicity d_1 and d_n respectively, and the degree of f is $\sum_{i=1}^n d_i$. Denote $D_i = d_i + d_{i+1}$, we have $5 \leq D_i \leq 2K$, where $1 \leq i \leq n-1$. Besides 0 and ∞, the rest $\sum_{i=1}^{n-1} D_i$ critical points of f are the solutions of

$$(-1)^p \, z \, \frac{f'(z)}{f(z)} = \sum_{i=1}^{n-1} \frac{(-1)^{n-i} D_i z^{D_i}}{z^{D_i} - a_i^{D_i}} + (-1)^n d_1 = 0. \tag{2.3}$$

For $1 \leq i \leq n-1$, let $\widetilde{CP}_i := \{\widetilde{w}_{i,j} = r_i a_i \exp(\pi i \frac{2j-1}{D_i}) : 1 \leq j \leq D_i\}$ be the collection of D_i points lying on the circle $\mathbb{T}_{r_i|a_i|}$ uniformly, where $r_i = \sqrt[D_i]{d_i/d_{i+1}}$. The following lemma shows that the $\sum_{i=1}^{n-1} D_i$ *free* critical points of f are very "close" to $\bigcup_{i=1}^{n-1} \widetilde{CP}_i$.

Lemma 2.3. *For every $\widetilde{w}_{i,j} \in \widetilde{CP}_i$, where $1 \leq i \leq n-1$ and $1 \leq j \leq D_i$, there exists $w_{i,j}$, which is a solution of (2.3), such that $|w_{i,j} - \widetilde{w}_{i,j}| < u^{\frac{2}{K}} |a_i|$. Moreover, $w_{i_1,j_1} = w_{i_2,j_2}$ if and only if $(i_1, j_1) = (i_2, j_2)$.*

Proof. Note that the right equation of (2.3) is equivalent to

$$(-1)^{n-i} \left(\frac{D_i z^{D_i}}{z^{D_i} - a_i^{D_i}} - d_i \right) + G_i(z) = 0, \tag{2.4}$$

where

$$G_i(z) = \sum_{1 \leq j \leq n-1, j \neq i} \frac{(-1)^{n-j} D_j z^{D_j}}{z^{D_j} - a_j^{D_j}} + (-1)^n d_1 + (-1)^{n-i} d_i. \tag{2.5}$$

Timing $(z^{D_i} - a_i^{D_i})/d_{i+1}$ on both sides of (2.4), where $1 \leq i \leq n-1$, we have

$$(-1)^{n-i} (z^{D_i} + d_i a_i^{D_i}/d_{i+1}) + (z^{D_i} - a_i^{D_i}) \, G_i(z)/d_{i+1} = 0. \tag{2.6}$$

Let $\Omega_i = \{z : |z^{D_i} + d_i a_i^{D_i}/d_{i+1}| \leq \varepsilon \, |a_i|^{D_i}\}$, where $\varepsilon = u^{\frac{2}{K}}$ and $1 \leq i \leq n-1$. For every $z \in \Omega_i$, since $\varepsilon \leq K^{-4}$ by Lemma 2.2(1), we have

$$K^{-1} < d_i/d_{i+1} - \varepsilon \leq |z/a_i|^{D_i} \leq d_i/d_{i+1} + \varepsilon < K - 1 < K. \tag{2.7}$$

This means that

$$K^{-1} < |a_i/z|^{D_i} < K \text{ and therefore } K^{-1} < |a_i/z|^5 < K. \tag{2.8}$$

If $1 \leq j < i$ and $z \in \Omega_i$, we have

$$|a_j/z|^{D_i} \leq |a_i/z|^{D_i} |a_{i-1}/a_i|^{D_i} < K u^{1+d_{i+1}/d_i} < 1. \tag{2.9}$$

Therefore, $|a_j/z| < 1$. By the similar argument, it can be shown that $|z/a_j| < 1$ if $i < j \leq n-1$ and $z \in \Omega_i$. If $1 \leq j < i$, by Lemma 2.2(1)(2) and (2.8), we have

$$|a_j/z|^{D_j} \leq |a_i/z|^5 |a_j/a_i|^5 < K \, \varepsilon^{5(i-j)/2} \leq K^{-9}. \tag{2.10}$$

Similarly, if $i < j \leq n-1$, we have

$$|z/a_j|^{D_j} \leq |z/a_i|^5 |a_i/a_j|^5 < K \, \varepsilon^{5(j-i)/2} \leq K^{-9}. \tag{2.11}$$

By definition, we have

$$\sum_{1 \leq j < i} (-1)^{n-j} D_j + (-1)^n d_1 + (-1)^{n-i} d_i = 0. \tag{2.12}$$

From (2.5), (2.10), (2.11) and (2.12), we have

$$\begin{aligned}|G_i(z)| &= \left|\sum_{1\le j<i}\frac{(-1)^{n-j}D_j}{1-(a_j/z)^{D_j}}+\sum_{i<j\le n-1}\frac{(-1)^{n-j-1}D_j(z/a_j)^{D_j}}{1-(z/a_j)^{D_j}}+(-1)^n d_1+(-1)^{n-i}d_i\right|\\ &\le 2\,K\left|\sum_{1\le j<i}\frac{(-1)^{n-j}(a_j/z)^{D_j}}{1-(a_j/z)^{D_j}}+\sum_{i<j\le n-1}\frac{(-1)^{n-j-1}(z/a_j)^{D_j}}{1-(z/a_j)^{D_j}}\right|\\ &< \frac{4K^2}{1-K^{-9}}\sum_{k=1}^{n-1}\varepsilon^{5k/2}<\frac{4K^2}{1-K^{-9}}\,\frac{\varepsilon^{5/2}}{1-\varepsilon^{5/2}}<5\,K^2\,\varepsilon^{5/2}\end{aligned}$$

since $\varepsilon^{5/2}\le K^{-10}$. This means that if $z\in\Omega_i$, we have

$$|z^{D_i}-a_i^{D_i}|\cdot|\,G_i(z)|/d_{i+1}<3\,K^3\,\varepsilon^{5/2}|a_i|^{D_i}<\varepsilon|a_i|^{D_i} \tag{2.13}$$

by (2.7) and Lemma 2.2(1).

From (2.6) and by Rouché's Theorem, there exists a solution $w_{i,j}$ of (2.3) such that $w_{i,j}\in\Omega_i$ for every $1\le j\le D_i$. In particular, $|w_{i,j}-\widetilde{w}_{i,j}|<\varepsilon|a_i|$ by the second statement of Lemma 2.1(2). Note that for $1\le i\le n-2$, we have

$$|a_{i+1}|-|a_i|-2\varepsilon|a_i|-2\varepsilon|a_{i+1}|>|a_{i+1}|(1-2\varepsilon-(1+2\varepsilon)K^{-2})>0. \tag{2.14}$$

By Lemma 2.2(1) and $r_i=\sqrt[D_i]{d_i/d_{i+1}}\le(K/2)^{1/5}$, we have,

$$\frac{r_i|a_i|\sin(\pi/D_i)}{\varepsilon|a_i|}\ge K^4(\frac{2}{K})^{1/5}\cdot\frac{2}{\pi}\cdot\frac{\pi}{2K}>K^2>1. \tag{2.15}$$

This means that $w_{i_1,j_1}=w_{i_2,j_2}$ if and only if $(i_1,j_1)=(i_2,j_2)$. The proof is completed. □

For $1\le i\le n-1$, let $CP_i:=\{w_{i,j}:1\le j\le D_i\}$ be the collection of D_i free critical points of f which lies closely to the circle $\mathbb{T}_{r_i|a_i|}$ and denote $CV_i=f(CP_i)$.

Lemma 2.4. *For every $1\le i\le n-1$, there exists an annular neighborhood A_i containing $CP_i\cup\mathbb{T}_{r_i|a_i|}\cup\mathbb{T}_{|a_i|}$, such that*

(1) If $p=1$, then $f(\overline{A}_i)\subset\mathbb{D}_s$ for odd $n-i$ and $f(\overline{A}_i)\subset\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}_K$ for even $n-i$. In particular, the set of critical values of f satisfies $\bigcup_{i=1}^{n-1}CV_i\subset\mathbb{D}_s\cup\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}_K$. The disks $\overline{\mathbb{D}}_s$ and $\overline{\mathbb{C}}\setminus\mathbb{D}_K$ lie in the Fatou set of f and $f^{-1}(\overline{\mathbb{A}}_{s,K})\subset\mathbb{A}_{s,K}$.

(2) If $p=0$, then $f(\overline{A}_i)\subset\mathbb{D}_s$ for even $n-i$ and $f(\overline{A}_i)\subset\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}_M$ for odd $n-i$, where $M=(2/s)^{1/d_n}$. In particular, the set of critical values of f satisfies $\bigcup_{i=1}^{n-1}CV_i\subset\mathbb{D}_s\cup\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}_M$. The disks $\overline{\mathbb{D}}_s$ and $\overline{\mathbb{C}}\setminus\mathbb{D}_M$ lie in the Fatou set of f and $f^{-1}(\overline{\mathbb{A}}_{s,M})\subset\mathbb{A}_{s,M}$.

Proof. Let $\varepsilon=u^{\frac{2}{K}}\le K^{-4}$ be the number appeared in Lemma 2.3. For every $1\le i\le n-1$, define the annulus

$$A_i=\{z:(\min\{r_i,1\}-2\varepsilon)|a_i|<|z|<(\max\{r_i,1\}+2\varepsilon)|a_i|\} \tag{2.16}$$

where $r_i=\sqrt[D_i]{d_i/d_{i+1}}$. Obviously, $A_i\supset CP_i\cup\mathbb{T}_{r_i|a_i|}\cup\mathbb{T}_{|a_i|}$. By the definition, we have

$$(2/K)^{\frac{1}{D_i}}\le\min\{r_i,1\}\le\max\{r_i,1\}\le(K/2)^{\frac{1}{D_i}}. \tag{2.17}$$

If $z\in\overline{A}_i$, we have

$$|a_i/z|\le\frac{1}{(2/K)^{\frac{1}{D_i}}-2\varepsilon}\le\frac{(K/2)^{\frac{1}{D_i}}}{1-2K^{-4}(K/2)^{1/5}}<(K/2)^{\frac{1}{D_i}}(1+4/K^{19/5}). \tag{2.18}$$

and

$$|z/a_i|\le(K/2)^{\frac{1}{D_i}}+2\varepsilon\le(K/2)^{\frac{1}{D_i}}+2/K^4<(K/2)^{\frac{1}{D_i}}(1+1/K^3). \tag{2.19}$$

This means that

$$|a_i/z|^5 < (K/2)^{\frac{5}{D_i}}(1+4/K^{19/5})^5 < (K/2)\,e^{20/K^{19/5}} < (K/2)\,e^{20/3^{19/5}} < 7K/10. \tag{2.20}$$

and also,

$$|z/a_i|^5 < (K/2)^{\frac{5}{D_i}}(1+1/K^3)^5 < (K/2)\,e^{5/K^3} < (K/2)\,e^{5/27} < 7K/10. \tag{2.21}$$

Moreover, similar to the argument of (2.20) and (2.21), we have

$$|a_i/z|^{d_i} + |z/a_i|^{d_{i+1}} < 7K/5. \tag{2.22}$$

Recall that $|a_i/a_{i+1}|^{d_{i+1}} = u$ for every $1 \le i \le n-2$ and $|a_{n-1}|^{d_n} = v$. Let $1 \le i_1 \le i_2 \le n-1$ and $p \in \{0,1\}$, we have

$$\begin{aligned}
\prod_{j=i_1}^{i_2} |a_j|^{(-1)^{n-j-p}D_j} &= |a_{i_1}|^{(-1)^{n-i_1-p}d_{i_1}}\,|a_{i_2}|^{(-1)^{n-i_2-p}d_{i_2+1}} \prod_{j=i_1}^{i_2-1}\left|\frac{a_j}{a_{j+1}}\right|^{(-1)^{n-j-p}d_{j+1}} \\
&= |a_{i_1}|^{(-1)^{n-i_1-p}d_{i_1}}\,|a_{i_2}|^{(-1)^{n-i_2-p}d_{i_2+1}}\,u^{\frac{(-1)^{n-i_1-p}-(-1)^{n-i_2-p}}{2}} \\
&= \begin{cases} (|a_1|^{d_1}u/v)^{(-1)^p} & \text{if } i_1 = 1 \text{ and } i_2 = n-1 \text{ is even} \\ (|a_1|^{-d_1}/v)^{(-1)^p} & \text{if } i_1 = 1 \text{ and } i_2 = n-1 \text{ is odd.} \end{cases}
\end{aligned} \tag{2.23}$$

By (1.2) and the second equation of (2.23), we have

$$\begin{aligned}
|f(z)| &= |z^{D_i} - a_i^{D_i}|^{(-1)^{n-i-p}}\,|z|^{(-1)^{n-p}d_1} \prod_{j=1}^{i-1}|z|^{(-1)^{n-j-p}D_j} \prod_{j=i+1}^{n-1}|a_j|^{(-1)^{n-j-p}D_j}\cdot Q_i(z) \\
&= |1-(z/a_i)^{D_i}|^{(-1)^{n-i-p}}\,|z/a_i|^{(-1)^{n-i-p+1}d_i}\,|a_{n-1}|^{(-1)^{1-p}d_n}\,u^{\frac{(-1)^{n-i-p}-(-1)^{1-p}}{2}}\cdot Q_i(z) \\
&= v^{(-1)^{1-p}}\,u^{\frac{(-1)^{n-i-p}-(-1)^{1-p}}{2}}\,|(a_i/z)^{d_i} - (z/a_i)^{d_{i+1}}|^{(-1)^{n-i-p}}\cdot Q_i(z) \\
&\begin{cases} \le v^{(-1)^{1-p}}\,u^{\frac{1-(-1)^{1-p}}{2}}\,(|a_i/z|^{d_i} + |z/a_i|^{d_{i+1}})\,Q_i(z) & \text{if } n-i-p \text{ is even} \\ \ge v^{(-1)^{1-p}}\,u^{\frac{-1-(-1)^{1-p}}{2}}\,(|a_i/z|^{d_i} + |z/a_i|^{d_{i+1}})^{-1}\,Q_i(z) & \text{if } n-i-p \text{ is odd,} \end{cases}
\end{aligned} \tag{2.24}$$

where

$$Q_i(z) = \prod_{j=1}^{i-1}\left|1-(a_j/z)^{D_j}\right|^{(-1)^{n-j-p}} \prod_{j=i+1}^{n-1}\left|1-(z/a_j)^{D_j}\right|^{(-1)^{n-j-p}}. \tag{2.25}$$

For $1 \le i \le n-1$, consider $z \in \overline{A}_i$. If $1 \le j < i$, by (2.20), we have

$$|a_j/z|^{D_j} \le |a_i/z|^5|a_j/a_i|^5 < 7K\,\varepsilon^{5(i-j)/2}/10 < K^{-9}. \tag{2.26}$$

If $i < j \le n-1$, then

$$|z/a_j|^{D_j} \le |z/a_i|^5|a_i/a_j|^5 < 7K\,\varepsilon^{5(i-j)/2}/10 < K^{-9}. \tag{2.27}$$

by (2.21). Since $e^x < 1+2x$ if $0 < x \le 1$ and $\varepsilon \le K^{-4}$, by (2.25)–(2.27), we have

$$Q_i(z) < \prod_{k=1}^{\infty}\left(1+7K\,\varepsilon^{5k/2}/5\right)^2 \le \exp\left(\frac{14\,K\,\varepsilon^{5/2}/5}{1-\varepsilon^{5/2}}\right) < 1+K^{-5} < 1.01. \tag{2.28}$$

and

$$Q_i(z) > \prod_{k=1}^{\infty}\left(1+7K\,\varepsilon^{5k/2}/5\right)^{-2} > 1/1.01 > 0.99. \tag{2.29}$$

For $p=1$, by Lemma 2.2(2)(3a), for every $1 \le i \le n-1$, if $|z| \le s$, we have

$$|z^{D_i}/a_i^{D_i}| \le |s/a_1|^{D_i}|a_1/a_i|^{D_i} \le (sK^{-3}/2)^{\frac{5}{K}}\,u^{\frac{5(i-1)}{K}}. \tag{2.30}$$

If we notice Lemma 2.2(1), then

$$\sum_{i=1}^{n-1} |z^{D_i}/a_i^{D_i}| \leq \frac{(sK^{-3}/2)^{\frac{5}{K}}}{1-u^{\frac{5}{K}}} \leq \frac{K^{\frac{10}{K}-10}}{1-K^{-10}} < 1/200. \tag{2.31}$$

For $p=0$, by Lemma 2.2(2)(4b), for every $1 \leq i \leq n-1$, if $|z| \leq s$, we have

$$|z^{D_i}/a_i^{D_i}| \leq |s/a_1|^{D_i}|a_1/a_i|^{D_i} \leq (u^{1/2}/2)^{\frac{5}{K}} u^{\frac{5(i-1)}{K}}. \tag{2.32}$$

By Lemma 2.2(1), then

$$\sum_{i=1}^{n-1} |z^{D_i}/a_i^{D_i}| \leq \frac{(u^{1/2}/2)^{\frac{5}{K}}}{1-u^{\frac{5}{K}}} \leq \frac{K^{-5}}{1-K^{-10}} < 1/200. \tag{2.33}$$

Since $(1+2|a|)^{-1} \leq |1+a|^{\pm 1} \leq 1+2|a|$ if $0 \leq |a| \leq 1/2$, by (2.31) and (2.33), we know that

$$\prod_{i=1}^{n-1} \left|1 - z^{D_i}/a_i^{D_i}\right|^{(-1)^{n-i-p}} \leq \prod_{i=1}^{n-1} \left(1+2|z/a_i|^{D_i}\right) < e^{1/100} < K. \tag{2.34}$$

Therefore,

$$\prod_{i=1}^{n-1} \left|1 - z^{D_i}/a_i^{D_i}\right|^{(-1)^{n-i-p}} \geq \prod_{i=1}^{n-1} \left(1+2|z/a_i|^{D_i}\right)^{-1} > e^{-1/100} > 1/K. \tag{2.35}$$

(1) We first consider the case $p=1$. If $n-i$ is odd, by (2.22), (2.24) and (2.28), if $z \in \overline{A}_i$ we have

$$|f(z)| \leq v \cdot (7K/5) \cdot 1.01 < 2Kv < s. \tag{2.36}$$

If $n-i$ is even, by (2.22), (2.24) and (2.29), for $z \in \overline{A}_i$ we have

$$|f(z)| \geq (v/u) \cdot (7K/5)^{-1} \cdot 0.99 > v/(2Ku) > K. \tag{2.37}$$

If n is odd, by Lemma 2.2(3a), (2.23) and (2.34), for every z such that $|z| \leq s$, we have

$$|f(z)| = |z|^{d_1} \prod_{i=1}^{n-1} |a_i|^{D_i(-1)^{n-i-1}} \prod_{i=1}^{n-1} \left|1 - \frac{z^{D_i}}{a_i^{D_i}}\right|^{(-1)^{n-i-1}} < |s/a_1|^{d_1} v u^{-1} \cdot 1.02 < s.$$

It follows that $f(\overline{\mathbb{D}}_s) \subset \mathbb{D}_s$ for odd n. If n is even and $|z| \leq s$, by Lemma 2.2(3b), (2.23) and (2.35), we have

$$|f(z)| = |a_1/z|^{d_1} v \prod_{i=1}^{n-1} \left|1 - \frac{z^{D_i}}{a_i^{D_i}}\right|^{(-1)^{n-i-1}} > |a_1/s|^{d_1} v/1.02 > K.$$

Therefore $f(\overline{\mathbb{D}}_s) \subset \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}_K$ for even n.

Note that f is very "close" to $z \mapsto z^{d_n}$ in the outside of $\mathbb{D}_K$ since $|a_i|^{D_i}$ is extremely small when it compares with those z such $|z| \geq K$, where $1 \leq i \leq n-1$. More specifically, by the completely similar arguments as (2.34)–(2.35), if $|z| \geq K$, then

$$|f(z)| \geq |z|^{d_n} \prod_{i=1}^{n-1} \left(1 + 2\frac{|a_i|^{D_i}}{|z|^{D_i}}\right)^{-1} > K. \tag{2.38}$$

This means that $f(\overline{\mathbb{C}} \setminus \mathbb{D}_K) \subset \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}_K$. Then we have $f^{-1}(\overline{\mathbb{A}}_{s,K}) \subset \mathbb{A}_{s,K}$ for every $n \geq 2$ (See Figure 1).

(2) Now we consider the case $p=0$. If $n-i$ is even, by (2.22), (2.24), (2.28) and Lemma 2.2(4a), if $z \in \overline{A}_i$ we have

$$|f(z)| \leq v^{-1} u \cdot (7K/5) \cdot 1.01 < 2Ku/v < s. \tag{2.39}$$

If $n-i$ is odd, by (2.22), (2.24), (2.29) and Lemma 2.2(4a), for $z \in \overline{A}_i$ we have

$$|f(z)| \geq v^{-1} \cdot (7K/5)^{-1} \cdot 0.99 > 1/(2Kv) > M, \tag{2.40}$$

where $M = (2/s)^{1/d_n}$.

If n is even, by Lemma 2.2(4b), (2.23) and (2.34), for each z such that $|z| \leq s$, we have

$$|f(z)| = |z|^{d_1} \prod_{i=1}^{n-1} |a_i|^{D_i(-1)^{n-i}} \prod_{i=1}^{n-1} \left|1 - \frac{z^{D_i}}{a_i^{D_i}}\right|^{(-1)^{n-i}} < |s/a_1|^{d_1} v^{-1} \cdot e^{1/100} < s.$$

It follows that $f(\overline{\mathbb{D}}_s) \subset \mathbb{D}_s$ for even n. If n is odd and $|z| \leq s$, by Lemma 2.2(4c), (2.23) and (2.35), we have

$$|f(z)| = |a_1/z|^{d_1} u v^{-1} \prod_{i=1}^{n-1} \left|1 - \frac{z^{D_i}}{a_i^{D_i}}\right|^{(-1)^{n-i}} \geq |a_1/s|^{d_1} u v^{-1} \cdot e^{-1/100} > M.$$

Therefore $f(\overline{\mathbb{D}}_s) \subset \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}_M$ for odd n.

If $|z| \geq M$, then

$$|f(z)| = |z|^{-d_n} \prod_{i=1}^{n-1} \left|1 - \frac{a_i^{D_i}}{z^{D_i}}\right|^{(-1)^{n-i}} \leq M^{-d_n} \prod_{i=1}^{n-1} \left(1 + \frac{2|a_i|^{D_i}}{|z|^{D_i}}\right) < 2M^{-d_n} = s. \tag{2.41}$$

This means that $f(\overline{\mathbb{C}} \setminus \mathbb{D}_M) \subset \mathbb{D}_s$. Then we have $f^{-1}(\overline{\mathbb{A}}_{s,M}) \subset \mathbb{A}_{s,M}$ for every $n \geq 2$. □

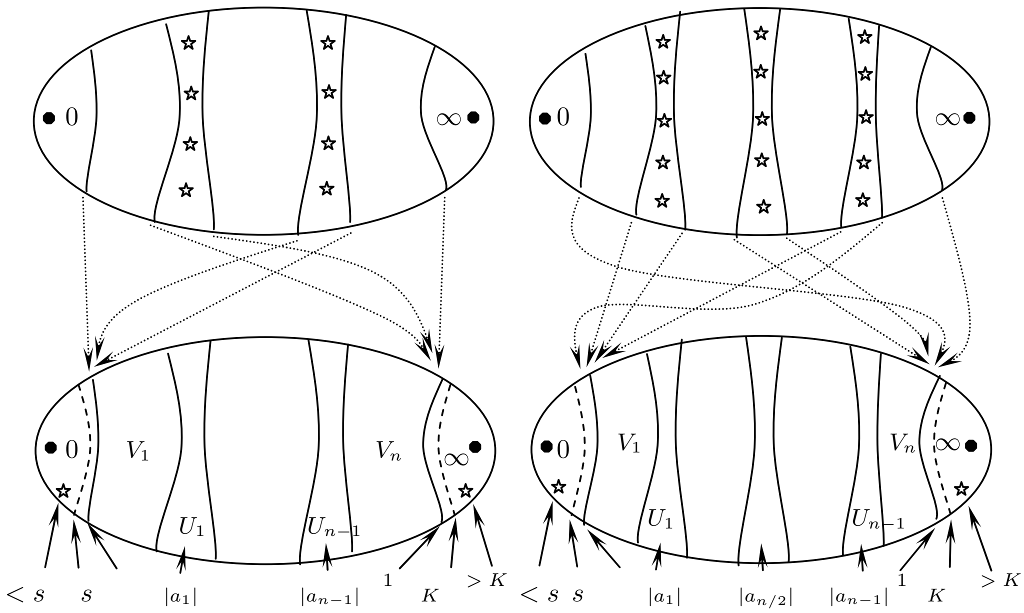

FIGURE 1. Sketch illustrating of the mapping relation of $f_{1,d_1,\cdots,d_n}$, where n is odd and even respectively (from left to right). The small pentagons denote the critical points and critical values, and the numbers showed at the bottom of the Figures denote the approximate coordinates.

Proof of Theorem 1.1. We only focus on the case $p = 1$ since the similar proof can be used to the case $p = 0$ by using Lemma 2.4(2). We also use f to denote $f_{1,d_1,\cdots,d_n}$ for simplicity. Let U_i be the component of $f^{-1}(D)$ containing a_i, where $D = \mathbb{D}_s$ if $n-i$ is odd and $D = \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}_K$ if $n-i$ is even. By Lemma 2.4(1), it follows that the set of critical points $CP_i \subset U_i$ and U_i is a connected domain containing the annulus A_i. Moreover, $U_i \cap U_{i+1} = \emptyset$ since $f(U_i) \cap f(U_{i+1}) = \emptyset$ by Lemma 2.4(1), where $1 \leq i < n-2$. This means that $U_i \cap U_j = \emptyset$ for different i, j. Suppose

that U_i has m_i boundary components. Since there are exactly D_i critical points in U_i and $f: U_i \to D$ is a branched covering with degree D_i, then the Riemann-Hurwitz formula tells us $\chi_{U_i} = 2 - m_i = D_i\chi_D - D_i = 0$, where χ denotes the Euler characteristic. This means that $m_i = 2$ and therefore U_i is an annulus surrounding the origin for every $1 \leq i \leq n-1$.

For $1 \leq i \leq n-2$, Let V_{i+1} be the annulus domain between U_i and U_{i+1}. It is easy to see $f: V_{i+1} \to \mathbb{A}_{s,K}$ is a covering map with degree d_{i+1}. Note that every component of $f^{-1}(\mathbb{A}_{s,K})$ is an annulus since $\mathbb{A}_{s,K}$ is double connected and contains no critical values. It follows that there exist two annuli V_1 and V_n, which lie between 0 and U_1, U_{n-1} and ∞ respectively, such that $f: V_1, V_n \to \mathbb{A}_{s,K}$ are covering maps with degree d_1 and d_n respectively. In fact, the restriction of f on ∂U_1 and ∂U_{n-1} has degree d_1 and d_n respectively and there are no critical points in V_1 and V_n (See Figure 1).

The Julia set of f is $J = \bigcap_{k\geq 0} f^{-k}(\mathbb{A}_{s,K})$. By the construction, the components of J are compact sets nested between 0 and ∞ since each inverse branch $f^{-1}: \mathbb{A}_{s,K} \to V_j$ is conformal for every $0 \leq j \leq n$. Since the component of J can not be a point and f is hyperbolic, every component of J is a Jordan curve (actually quasicircle) by Theorem 1.2 in [PT]. The dynamics on the set of Julia components of f is isomorphic to the one-sided shift on n symbols $\Sigma_n := \{0, 1, \cdots, n-1\}^{\mathbb{N}}$. In particular, J is homeomorphic to $\Sigma_n \times \mathbb{S}^1$, which is a Cantor set of circles as desired (See Figure 2 for example). This completes the proof of Theorem 1.1. □

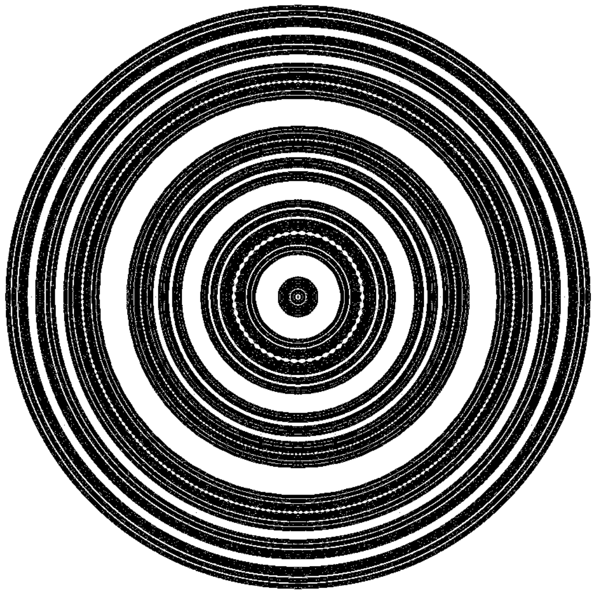

FIGURE 2. The Julia set of $f_{1,5,5,5,5}$, which is clearly a Cantor set of circles, where the parameter s is chosen small enough.

Remark: Since f is hyperbolic, the Julia set of f is also a Cantor set of circles if we disturb some a_i gently, where $1 \leq i \leq n-1$. In the first version of our manuscript of this paper, only $d_i = n+1$ for every $1 \leq i \leq n$ was considered. In this case, it was shown that for every $n \geq 2$ and $1 \leq i \leq n-1$, if $|a_{n-i}| = (\frac{n}{n+1})^{i-1}s^i$ for $0 < s \leq 1/10$, then the Julia set of $f_{1,n+1,\cdots,n+1}$ is a Cantor set of circles.

Theorem 2.5. *Suppose that a_i is chosen like in Theorem 1.1 such that the Julia set of $f_{p,d_1,\cdots,d_n}$ is a Cantor set of circles for $n \geq 3$, then $f_{p,d_1,\cdots,d_n}$ is not topologically conjugate to any McMullen maps on their corresponding Julia sets.*

Proof. Since the dynamics on the set of Julia components of $f_{p,d_1,\cdots,d_n}$ is conjugated to the one-sided shift on n symbols $\Sigma_n := \{0, 1, \cdots, n-1\}^{\mathbb{N}}$ and in particular, the set of Julia components

of g_η is isomorphic to the one-sided shift on only 2 symbols $\Sigma_2 := \{0,1\}^{\mathbb{N}}$. This means that $f_{p,d_1,\cdots,d_n}$ can not be topologically conjugate to g_η on their corresponding Julia sets if $n \geq 3$. $\square$

3. Topological conjugacy between the Cantor circles Julia sets

In this section, we show that for any given rational map whose Julia set is a Cantor set of circles, then there exists a map $f_{p,d_1,\cdots,d_n}$ in (1.2) such these two rational maps are topological conjugate on their corresponding Julia sets.

Lemma 3.1. *If f is a rational map whose Julia set is a Cantor set of circles. Then there exists no critical points on $J(f)$.*

Proof. Suppose there exists a Julia component J_0 of f containing a critical point c_0 of f with multiplicity d. Then f is not one to one in any small neighborhood of c_0. It is known $f(J_0)$ is a Julia component containing $f(c_0)$ [Be, Lemma 5.7.2]. Choose a small topological disk neighborhood U of $f(c_0)$ such that $U \cap f(J_0)$ is a simple curve. The component of $f^{-1}(U)$ containing c_0 is mapped onto U in the manner of $d+1$ to one. Note that the component J' of $f^{-1}(U \cap f(J_0))$ containing c_0 is connected and contained in J_0. However, J' possesses star-like structure and hence is not a simple curve. This contradicts to the assumption that J_0 is a Jordan closed curve since $J(f)$ is a Cantor set of circles. $\square$

We say that a compact set $A \subset \overline{\mathbb{C}}$ *separates* 0 and ∞ if 0 and ∞ lie in the two different components of $\overline{\mathbb{C}} \setminus A$ respectively. Let A and B be two disjoint compact sets both separates 0 and ∞ respectively. We say $A \prec B$ if A is contained in the component of $\overline{\mathbb{C}} \setminus B$ which contains 0. Let A be an annulus separating 0 and ∞, we use $\partial_- A$ and $\partial_+ A$ to denote the two components of the boundary of A such that $\partial_- A \prec \partial_+ A$.

Theorem 3.2. *Let f be a rational map whose Julia set is a Cantor set of circles. Then there exist $p \in \{0,1\}$, positive integers $n \geq 2$ and $d_1, \cdots, d_n$ satisfying $\sum_{i=1}^n \frac{1}{d_i} < 1$ such that f is topologically conjugate to $f_{p,d_1,\cdots,d_n}$ on their corresponding Julia sets.*

Proof. Let $J(f)$ be the Julia set of f which is a Cantor set of circles, then every periodic Fatou component of f must be attracting or parabolic by Lemma 3.1. We only prove the attracting (hyperbolic) case in detail and explain the parabolic case by using the work of Cui [Cui].

In the following, we suppose that f is hyperbolic. There exist exactly two simply connected Fatou components of f and all other Fatou components are annuli. Let $\mathcal{D}$ and $\mathcal{A}$ be the collection of simply and doubly connected Fatou components of f respectively. We claim that $f(\mathcal{D}) \subset \mathcal{D}$ and there exists an integer $k \geq 1$ such that $f^{\circ k}(A) \in \mathcal{D}$ for every $A \in \mathcal{A}$. The assertion $f(\mathcal{D}) \subset \mathcal{D}$ is obvious since the image of a simply connected Fatou component under a rational map is again simply connected. If $f(A_1) = A_2$, where $A_1, A_2 \in \mathcal{A}$, then there exists no critical points in A_1 by Riemann-Hurwitz's formula. This means that each $A \in \mathcal{A}$ can not be periodic since the cycle of every periodic attracting Fatou component must contain at least one critical point. On the other hand, by Sullivan's theorem, the Fatou components of a rational map can not be wandering. This completes the proof of claim.

Up to a Mobius transformation, we can assume that 0 and ∞, respectively, are belong to the two simply connected Fatou components of f, which are denoted by D_0 and D_∞. Namely, $\mathcal{D} = \{D_0, D_\infty\}$. Since $f(\mathcal{D}) \subset \mathcal{D}$, without loss of generality, we suppose that $f(D_0) = D_0$ and $f(D_\infty) = D_\infty$. Let $f^{-1}(D_0) = D_0 \cup A_1 \cup \cdots \cup A_m$, where $A_1, \cdots, A_m$ are m annuli separating 0 and ∞ such that $A_i \prec A_{i+1}$ for every $1 \leq i \leq m-1$. It is easy to see $m \geq 1$. Otherwise, D_0 is completely invariant, then $J(f) = \partial D_0$ which contradicts to the assumption that $J(f)$ is a Cantor set of circles.

Suppose that $\deg(f|_{D_0} : D_0 \to D_0) = d_1$ and $\deg(f|_{\partial_- A_i} : \partial_- A_i \to \partial D_0) = d_{2i}$ and $\deg(f|_{\partial_+ A_i} : \partial_+ A_i \to \partial D_0) = d_{2i+1}$ for $1 \leq i \leq m$. It follows that $\deg(f) = \sum_{j=1}^{2m+1} d_j$. Let W_1 be the annular domain between D_0 and A_1 and W_i be the annular domain between A_{i-1} and A_i, where $2 \leq i \leq m$. We have $f(W_i) = \overline{\mathbb{C}} \setminus \overline{D}_0$ and $\deg(f|_{W_i} : W_i \to \overline{\mathbb{C}} \setminus \overline{D}_0) = d_{2i-1} + d_{2i}$. This means

that there exists at least one Fatou component $B_i \subsetneq W_i$ such that $f(B_i) = D_\infty$. If there exists $B_i' \neq B_i$ such that $B_i' \subsetneq W_i$ and $f(B_i') = D_\infty$, there must exist one component of $f^{-1}(D_0)$ in W_i, which contradicts to the assumption $A_1 \cup \cdots \cup A_m$ is the collection of all annular components of $f^{-1}(D_0)$. So there exists exactly one Fatou component $B_i \subsetneq W_i$ such that $f(B_i) = D_\infty$ and $\deg(f|_{B_i} : B_i \to D_\infty) = d_{2i-1} + d_{2i}$. Similar argument can be used to show that D_∞ is the only component of $f^{-1}(D_\infty)$ lying in the unbounded component of $\overline{\mathbb{C}} \setminus A_m$ which can be mapped onto D_∞. Therefore, $f^{-1}(D_\infty) = B_1 \cup \cdots \cup B_m \cup D_\infty$ and $\deg(f|_{D_\infty}) = d_{2m+1}$ since $\deg(f) = \sum_{j=1}^{2m+1} d_j$. Denote $\overline{\mathbb{C}} \setminus (D_0 \cup D_\infty)$ by E. The preimage $f^{-1}(E)$ consists of $2m+1$ annuli components $E_1, \cdots, E_{2m+1}$ such that $E_i \prec E_{i+1}$ for $1 \leq i \leq 2m$. The map $f : E_i \to E$ is a unramified covering map with degree d_i, where $1 \leq i \leq 2m+1$ (See Figure 3).

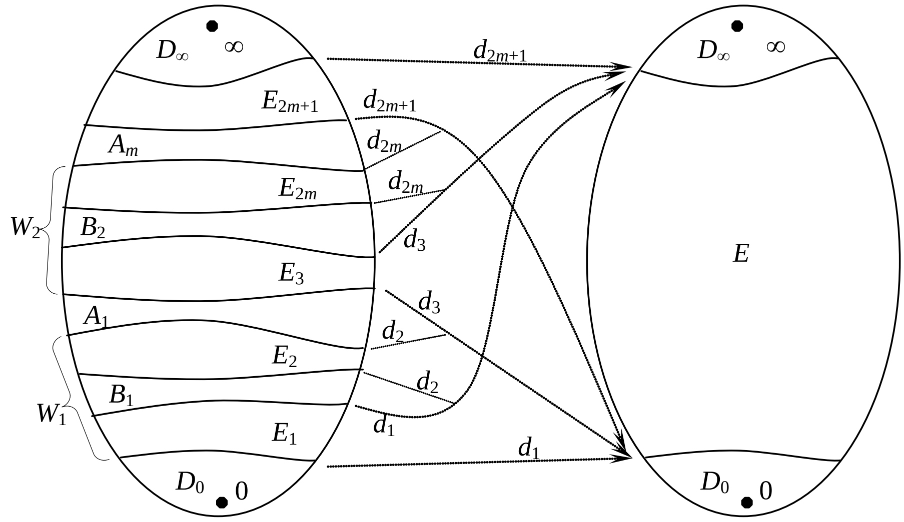

FIGURE 3. Sketch illustrating of the mapping relation of f, where d_i, $1 \leq i \leq 2m+1$ denote the degrees of the restriction of f on the boundaries of Fatou components.

Let $n = 2m+1$ and $p = 1$. The assertion $\sum_{i=1}^n 1/d_i < 1$ follows from Grötzsch's modulus inequality since each E_i is essentially contained in E and $\mathrm{mod}(E_i) = \mathrm{mod}(E)/d_i$. In the following, we will construct a quasiconformal map $\phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ which conjugates the dynamics on the Julia set of f to that of $f_{1,d_1,\cdots,d_n}$.

For simplicity, we denote $f_{1,d_1,\cdots,d_n}$ by F. Note that $F(0) = 0$ and $F(\infty) = \infty$. There exist two simply connected Fatou components D_0' and D_∞', both are invariant under F such that $0 \in D_0'$ and $\infty \in D_\infty'$. From the proof of Theorem 1.1, we know that $F^{-1}(D_0') = D_0' \cup A_1' \cup \cdots \cup A_m'$, where $A_1', \cdots, A_m'$ are m annuli separating 0 and ∞ such that $A_i' \prec A_{i+1}'$ for every $1 \leq i \leq m-1$. Moreover, $\deg(F|_{D_0'} : D_0' \to D_0') = d_1$ and $\deg(F|_{\partial_- A_i'} : \partial_- A_i' \to \partial D_0') = d_{2i}$ and $\deg(F|_{\partial_+ A_i'} : \partial_+ A_i' \to \partial D_0') = d_{2i+1}$ for $1 \leq i \leq m$. Let W_1' be the annular domain between D_0' and A_1' and W_i' be the annular domain between A_{i-1}' and A_i', where $2 \leq i \leq m$. There exists exactly one Fatou component $B_i' \subsetneq W_i'$ such that $F(B_i') = D_\infty'$ and $\deg(F|_{B_i'} : B_i' \to D_\infty') = d_{2i-1} + d_{2i}$. We have $F^{-1}(D_\infty') = B_1' \cup \cdots \cup B_m' \cup D_\infty'$ and $\deg(F|_{D_\infty'}) = d_{2m+1}$. Similarly, let $E' := \overline{\mathbb{C}} \setminus (D_0' \cup D_\infty')$. There exist $2m+1$ annuli components $E_1', \cdots, E_{2m+1}'$ of $F^{-1}(E')$ such that $E_i' \prec E_{i+1}'$ for $1 \leq i \leq 2m$. The map $F : E_i' \to E'$ is a covering with degree d_i, where $1 \leq i \leq 2m+1$.

By a quasiconformal surgery, it can be seen that $\partial D_0, \partial D_\infty, \partial D_0', \partial D_\infty'$ and their preimages are all quasicircles and the dilatation is bounded by a fixed constant. There exists a quasiconformal mapping $\phi_0 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that $\phi_0(D_0) = D_0'$ and $\phi_0(D_\infty) = D_\infty'$ hence $\phi_0(\partial D_0) = \partial D_0'$ and $\phi_0(\partial D_\infty) = \partial D_\infty'$. Moreover, ϕ_0 can be chosen such that $\phi_0 \circ f = F \circ \phi_0$ on $\partial D_0 \cup \partial D_\infty$.

Now we construct a lift $\phi_{E_1} : E_1 \to E_1'$ of $\phi_0 : E \to E'$ as follows. For every $z \in E_1 \setminus \partial_- E_1$, we choose a simple curve $\gamma : [0,1] \to E$ such that $\gamma(1) = f(z)$ and $\gamma(0) = w \in \partial_- E$. Since

$f : E_1 \to E$ is a covering map, there exists a unique lift $\widetilde{\gamma} : [0,1] \to E_1$ of γ such that $\widetilde{\gamma}(1) = z$ and $\widetilde{w} := \widetilde{\gamma}(0) \in \partial_- E_1$. Similarly, since $F : E_1' \to E'$ is a covering map, there exists a unique lift $\alpha : [0,1] \to E_1'$ of $\phi_0(\gamma) : [0,1] \to E'$ such that $\alpha(0) = \phi_0(\widetilde{w})$ since $\phi_0 \circ f = F \circ \phi_0$ on $\partial D_0 = \partial_- E_1$. Define $\phi_{E_1}(z) := \alpha(1)$. We know that $\phi_0 \circ f = F \circ \phi_{E_1}$ on E_1 and $\phi_{E_1} : E_1 \to E_1'$ is quasiconformal since f, F are both holomorphic covering maps with degree d_1 and $\phi_0 : E \to E'$ is quasiconformal. Now some parts of $\phi_1 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ are defined as follows: $\phi_1|_{\overline{D}_0} = \phi_0|_{\overline{D}_0}$, $\phi_1|_{\overline{D}_\infty} = \phi_0|_{\overline{D}_\infty}$ and $\phi_1|_{E_1} = \phi_{E_1}$. Then, $\phi_1 \circ f = F \circ \phi_1$ on ∂E_1. Similarly, there exists a unique quasiconformal mapping $\phi_{E_{2m+1}} : E_{2m+1} \to E_{2m+1}'$, which is the lift of $\phi_0 : E \to E'$ such that $\phi_0 \circ f = F \circ \phi_{E_{2m+1}}$ on E_{2m+1}. Define $\phi_1|_{E_{2m+1}} = \phi_{E_{2m+1}}$. Then, $\phi_1 \circ f = F \circ \phi_1$ on ∂E_{2m+1}.

Unlike the cases of E_1 and E_{2m+1}, the lift $\phi_{E_i} : E_i \to E_i'$ of $\phi_0 : E \to E'$ is exist but not unique for $2 \le i \le 2m$. We first show the existence of ϕ_{E_i}. Without loss of generality, suppose that i is even. Since $f : \partial_- E_i \to \partial D_\infty$ and $F : \partial_- E_i' \to \partial D_\infty'$ are both covering mappings with degree d_i, there exists a lift (not unique) $\phi_{E_i} : \partial_- E_i \to \partial_- E_i'$ of $\phi_0 : \partial D_\infty \to \partial D_\infty'$ such that $\phi_0 \circ f = F \circ \phi_{E_i}$ on $\partial_- E_i$. By using the same method of defining ϕ_{E_1}, there exists a unique lift of $\phi_0 : E \to E'$ defined from E_i to E_i', which we denote also by ϕ_{E_i} such that $\phi_0 \circ f = F \circ \phi_{E_i}$ on E_i. Note that $\phi_{E_i} : E_i \to E_i'$ is quasiconformal. Define $\phi_1|_{E_i} = \phi_{E_i}$. Then, $\phi_0 \circ f = F \circ \phi_1$ on $\bigcup_{i=1}^{2m+1} E_i$ and $\phi_1 \circ f = F \circ \phi_1$ on $\bigcup_{i=1}^{2m+1} \partial E_i$.

In order to unify the notations, let $D_{2i-1} := B_i$ and $D_{2i} := A_i$ for $1 \le i \le m$. Then we have $D_i \prec D_j$ for $1 \le i < j \le 2m$. We need to define ϕ_1 on $\bigcup_{i=1}^{2m} D_i$. For every D_i, where $1 \le i \le 2m$, its two boundary components $\partial_+ E_i$ and $\partial_- E_{i+1}$ are both quasicircles. Since ϕ_{E_i} and $\phi_{E_{i+1}}$ are both quasiconformal mappings, the map $\phi_1|_{\partial_+ E_i \cup \partial_- E_{i+1}}$ has a quasiconformal extension $\phi_{D_i} : \overline{D}_i \to \overline{D}_i'$ such that $\phi_{D_i}(D_i) = D_i'$. Now we obtain a quasiconformal mapping $\phi_1 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ defined as $\phi_1|_{E_i} := \phi_{E_i}$, $\phi_1|_{D_j} = \phi_{D_j}$ and $\phi_1|_{D_0 \cup D_\infty} = \phi_0$, where $1 \le i \le 2m+1$ and $1 \le j \le 2m$.

Next, we define ϕ_2. First, let $\phi_2|_{D_j} = \phi_1$ for $j \in \{0, 1, \cdots, 2m, \infty\}$. Then we lift $\phi_1 : E \to E'$ in appropriate ways to obtain $\phi_2 : E_i \to E_i'$ for $1 \le i \le 2m+1$. Finally, we check the continuity of the resulting map $\phi_2 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$. Now let us make it precisely. In order to make sure the continuity of ϕ_2 on $D_0 \cup E_1$, we need to have $\phi_2|_{\partial_- E_1} = \phi_1$. Then there exists only one way to lift $\phi_1 : E \to E'$ to obtain $\phi_2 : E_1 \to E_1'$. Note that $\phi_2|_{D_1} = \phi_1$, we need to check the continuity of ϕ_2 at the boundary $\partial_+ E_1$. In fact, $\phi_0|_E$ and $\phi_1|_E$ are homotopic to each other and $\phi_1|_{\partial E} = \phi_0|_{\partial E}$, it follows that $\phi_2|_{\partial_+ E_1} = \phi_1|_{\partial_+ E_1}$ since $\phi_2|_{\partial_- E_1} = \phi_1|_{\partial_- E_1}$. This means that ϕ_2 is continuous on $\partial_+ E_1$. Similarly, we can lift $\phi_1 : E \to E'$ to obtain $\phi_2 : E_i \to E_i'$ for $2 \le i \le 2m+1$ and guarantee the continuity of ϕ_2. Above all, the map $\phi_2 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ satisfies (1) ϕ_2 is quasiconformal and the dilatation $K(\phi_2) = K(\phi_1)$; (2) $\phi_2|_{f^{-1}(D_0 \cup D_\infty)} = \phi_1$; (3) $\phi_1 \circ f = F \circ \phi_2$ on $\bigcup_{i=1}^{2m+1} E_i$ and hence $\phi_2 \circ f = F \circ \phi_2$ on $f^{-2}(\partial D_0 \cup \partial D_\infty)$.

Suppose we have obtained ϕ_k for some $k \ge 1$, then ϕ_{k+1} can be defined completely similarly as the process of the derivation of ϕ_2 from ϕ_1. Inductively, we can obtain a sequence of quasiconformal mappings $\{\phi_k\}_{k \ge 0}$ such that (1) $K(\phi_k) = K(\phi_1) \ge K(\phi_0)$ for $k \ge 1$; (2) $\phi_{k+1}(z) = \phi_k(z)$ for $z \in f^{-k}(D_0 \cup D_\infty)$; (3) $\phi_k \circ f = F \circ \phi_k$ on $f^{-k}(\partial D_0 \cup \partial D_\infty)$. This means that $\{\phi_k\}_{k \ge 0}$ forms a normal family. Take a convergent subsequence of $\{\phi_k\}_{k \ge 0}$ whose limit we denote by ϕ_∞, then ϕ_∞ is a quasiconformal mapping satisfying $\phi_\infty \circ f = F \circ \phi_\infty$ on $\bigcup_{k \ge 0} f^{-k}(\partial D_0 \cup \partial D_\infty)$. Moreover, $K(\phi_\infty) \le K(\phi_1)$. Since ϕ_∞ is continuous, $\phi_\infty \circ f = F \circ \phi_\infty$ holds on the closure of $\bigcup_{k \ge 0} f^{-k}(\partial D_0 \cup \partial D_\infty)$, which is the Julia set of f. Therefore $\phi = \phi_\infty$ is the quasiconformal mapping we want to find which conjugates f to F on their corresponding Julia sets. This ends the proof of case $f(D_0) = D_0$ and $f(D_\infty) = D_\infty$.

The other three cases: (1) $f(D_0) = D_\infty$, $f(D_\infty) = D_\infty$; (2) $f(D_0) = D_\infty$, $f(D_\infty) = D_0$; and (3) $f(D_0) = D_0$, $f(D_\infty) = D_0$ can be proved completely similarly.

If one or both of the components D_0 and D_∞ are parabolic, there exists a perturbation f_ε of f such that f_ε is hyperbolic and the dynamics of f_ε is topologically conjugate to that of f on their corresponding Julia sets [Cui]. Then f has a "model" in (1.2) since f_ε always does. This ends the proof of Theorem 3.2 and hence Theorem 1.2. □

From the proof of Theorem 3.2 in the hyperbolic case, we have following immediate corollary.

Corollary 3.3. *If the parameters a_i are chosen like in Theorem 1.1, where $1 \leq i \leq n-1$, then each Julia component of $f_{p,d_1,\cdots,d_n}$ is a quasicircle.*

4. Quasisymmetric geometry of the Cantor circles

Recall that the *conformal dimension* $\mathrm{confdim}(X)$ of a metric space X is the infimum of the Hausdorff dimensions of all metric spaces which are quasisymmetrically equivalent to X.

Proof of Theorem 1.3. From the proof of Theorem 1.1, we know that the combinatorics of $f_{p,d_1,\cdots,d_n}$ is determined by data $\mathcal{D} := (d_1, \cdots, d_n) \in \mathbb{N}^n$ in the sense of Haïssinsky and Pilgrim [HP, §2]. By Propositions 1.1 and 2.2 in [HP], the conformal dimension of the Julia set of $f_{p,d_1,\cdots,d_n}$ is $\mathrm{confdim}(J_{p,d_1,\cdots,d_n}) = 1 + \alpha_{p,d_1,\cdots,d_n}$, where $\alpha_{p,d_1,\cdots,d_n}$ is the unique positive root of $\sum_{i=1}^n d_i^{-\alpha_{p,d_1,\cdots,d_n}} = 1$. In particular, if $d_i = n+1$ for every $1 \leq i \leq n$, then $\alpha_n := \alpha_{p,d_1,\cdots,d_n} = \log(n)/\log(n+1)$. This means that $m \neq n$ is equivalent to $\alpha_m \neq \alpha_n$.

Lemma 4.1. *If $n \geq 3$, then $x = \log(n)/\log(n+1)$ is not the solution of*

$$k^{-x} + l^{-x} = 1, \tag{4.1}$$

where $k, l \geq 2$ are two integers such that $1/k + 1/l < 1$.

Proof. Without loss of generality, we suppose that $2 \leq k \leq l$, then $1/k^x \geq 1/l^x$, where $x = \log(n)/\log(n+1)$. If $n \geq 3$, then

$$\frac{1}{k^x} + \frac{1}{l^x} \leq \frac{1}{2^{\log 3/\log 4}} + \frac{1}{3^{\log 3/\log 4}} = 0.9960381127\cdots < 1$$

since $\log(n-1)/\log(n) < \log(n)/\log(n+1)$. This completes the proofs of Lemma 4.1 and Theorem 1.3. □

Proofs of Corollaries 1.4 and 1.5. They are immediate corollaries of Theorem 1.3 if we notice that the conformal dimension is an invariant of the quasisymmetric class of a metric space. □

5. Non-hyperbolic rational maps whose Julia sets are Cantor circles

The rational maps

$$P_\lambda(z) = \frac{\frac{1}{n}((1+z)^n - 1) + \lambda^{m+n} z^{m+n}}{1 - \lambda^{m+n} z^{m+n}} \tag{5.1}$$

where $\lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $m, n \geq 2$ are both positive integers satisfying $1/m + 1/n < 1$ can be seen as a perturbation of the parabolic polynomial

$$\widetilde{P}(z) = \frac{(1+z)^n - 1}{n}. \tag{5.2}$$

Note that $\widetilde{P}$ has a parabolic fixed point at the origin with multiplier 1 and critical point -1 with multiply $n-1$. This means that there exists only one bounded and hence simply connected Fatou component of $\widetilde{P}$ in which all points are attracted to the origin. In particular, the Julia set of $\widetilde{P}$ is a Jordan curve with infinite cusps.

We hope that some properties of $\widetilde{P}$ stated above can be also hold for P_λ when λ is small. But obviously, there are lots of differences between P_λ and $\widetilde{P}$. The degree of P_λ is $m+n$ and $P_\lambda(\infty) = -1$. There are $2(m+n)-2$ critical points of P_λ: $m-1$ at ∞, $n-1$ are very close to -1 and the rest $m+n$ critical points lie nearby the circle $\mathbb{T}_{r_0/|\lambda|}$, where $r_0 = \sqrt[m+n]{n/m}$ (See Lemma 5.3). In fact, we will see that P_λ can be viewed as a "parabolic" McMullen map at the end of this section since P_λ is conjugate to some g_η on their corresponding Julia sets.

Firstly, we show that the fixed parabolic Fatou component of $\widetilde{P}$ contains the Euclidean disk $\mathbb{D}(-\frac{3}{4}, \frac{3}{4})$ for every $n \geq 2$ and P_λ maps $\mathbb{D}(-\frac{3}{4}, \frac{3}{4})$ into itself if λ is small enough.

Lemma 5.1. (1) *For every* $n\geq 2$, $\widetilde{P}(\overline{\mathbb{D}}(-\frac{3}{4},\frac{3}{4}))\subset\mathbb{D}(-\frac{3}{4},\frac{3}{4})\cup\{0\}$.

(2) *If* $0<|\lambda|<1/(3n)$, *then* $P_\lambda(\overline{\mathbb{D}}(-\frac{3}{4},\frac{3}{4}))\subset\mathbb{D}(-\frac{3}{4},\frac{3}{4})\cup\{0\}$. *In particular,* $\mathbb{D}(-\frac{3}{4},\frac{3}{4})$ *lies in the parabolic Fatou component of* P_λ *with parabolic fixed point* 0.

Proof. If $z\in\overline{\mathbb{D}}(-\frac{3}{4},\frac{3}{4})$, then $|\widetilde{P}(z)+1/n|=|1+z|^n/n\leq 1/n$. In particular, the inequality sign can be replaced by equality if and only if $z=0$. This ends the proof of (1).

The proof of (2) will be divided into two cases: $|z|$ is small and not too small. For every $z=-\frac{3}{4}+\frac{3}{4}e^{i\theta}\in\partial\mathbb{D}(-\frac{3}{4},\frac{3}{4})$, where $-\pi<\theta\leq\pi$, we have $|1+\widetilde{P}(z)|\leq 5/2$ by (1) and $|\lambda z|^{m+n}<1/2$ since $|\lambda|<1/(3n)$. This means that

$$|P_\lambda(z)-\widetilde{P}(z)|=\left|\frac{\lambda^{m+n}z^{m+n}(1+\widetilde{P}(z))}{1-\lambda^{m+n}z^{m+n}}\right|\leq 5|\lambda z|^{m+n}. \tag{5.3}$$

Since $|z|=\frac{3}{4}|1-e^{i\theta}|=\frac{3}{4}|e^{-i\theta/2}-e^{i\theta/2}|=\frac{3}{2}|\sin\frac{\theta}{2}|\leq\frac{3}{4}|\theta|$ and $|\lambda|<1/(3n)$, we have

$$|P_\lambda(z)-\widetilde{P}(z)|\leq 5\,(|\theta|/(4n))^{m+n}. \tag{5.4}$$

On the other hand, since $|\sin\theta|\geq\frac{2}{\pi}|\theta|$ if $|\theta|\leq\frac{\pi}{2}$, we have

$$\begin{aligned}|\widetilde{P}(z)+3/4|&=\left|\frac{(\frac{1}{4}+\frac{3}{4}e^{i\theta})^n-1}{n}+\frac{3}{4}\right|\leq\frac{|\frac{1}{4}+\frac{3}{4}e^{i\theta}|^n-1}{n}+\frac{3}{4}\\&=\frac{(1-\frac{3}{4}\sin^2\frac{\theta}{2})^{n/2}-1}{n}+\frac{3}{4}\leq\frac{(1-\frac{3\theta^2}{4\pi^2})^{n/2}-1}{n}+\frac{3}{4}.\end{aligned} \tag{5.5}$$

If $|\theta|<2\pi/n$, then $\frac{3\theta^2}{4\pi^2}<\frac{2}{n}$. By Lemma 2.1(3), we have

$$|\widetilde{P}(z)+3/4|\leq-\frac{\frac{n}{2}\cdot\frac{3\theta^2}{4\pi^2}}{3n}+\frac{3}{4}=\frac{3}{4}-\frac{\theta^2}{8\pi^2}. \tag{5.6}$$

Therefore, combine (5.4) and (5.6), it follows that if $|\theta|<2\pi/n$, then

$$|P_\lambda(z)+3/4|\leq|\widetilde{P}(z)+3/4|+|P_\lambda(z)-\widetilde{P}(z)|\leq\frac{3}{4}-\frac{\theta^2}{8\pi^2}+5\,(\frac{|\theta|}{4n})^{m+n}\leq 3/4. \tag{5.7}$$

If $2\pi/n\leq|\theta|\leq\pi$, from (5.5) and (5.6), we know that

$$|\widetilde{P}(z)+3/4|\leq\frac{3}{4}-\frac{1}{2n^2}. \tag{5.8}$$

From (5.4) and (5.8), it follows that if $2\pi/n\leq|\theta|\leq\pi$, then

$$|P_\lambda(z)+3/4|\leq\frac{3}{4}-\frac{1}{2n^2}+5\,(\frac{|\theta|}{4n})^{m+n}<3/4. \tag{5.9}$$

Whatever, we have shown that $|P_\lambda(z)+\frac{3}{4}|\leq\frac{3}{4}$ for every $z\in\partial\mathbb{D}(-\frac{3}{4},\frac{3}{4})$ and $|P_\lambda(z)+\frac{3}{4}|=\frac{3}{4}$ if and only if $z=0$. The proof is completed. □

As the procedure in Section 2, now we locate the free critical points of P_λ. By a direct calculation, the bounded $m+2n-1$ critical points of P_λ are the solution of

$$(1+z)^{n-1}+\lambda^{m+n}z^{m+n-1}\{(1+m/n)[(1+z)^n+n-1]-z(1+z)^{n-1}\}=0. \tag{5.10}$$

Lemma 5.2. *If* $0<|\lambda|<1/(3n)$, *then there are* $n-1$ *critical points of* P_λ *in* $\mathbb{D}(-1,|\lambda|)\subsetneq\mathbb{D}(-\frac{3}{4},\frac{3}{4})$.

Proof. If $|z+1|\leq|\lambda|<\frac{1}{3n}$, then $|z|\cdot|1+z|^{n-1}\leq(1+|\lambda|)|\lambda|^{n-1}<1$ and

$$(1+m/n)\,|(1+z)^n+n-1|\leq(1+m/n)(|\lambda|^n+n-1)<m+n. \tag{5.11}$$

This means that if $|z+1| \leq |\lambda|$, then

$$\begin{aligned}
&\left|\lambda^{m+n}z^{m+n-1}\{(1+m/n)[(1+z)^n+n-1]-z(1+z)^{n-1}\}\right| \\
<\ & |\lambda|^{n-1}\cdot|\lambda z|^{m-1}|\lambda|^2|z|^n(m+n+1) < |\lambda|^{n-1}\cdot(2n)^{1-m}(9n^2)^{-1}e^{1/3}(m+n+1) \\
<\ & |\lambda|^{n-1}\cdot(m+n-1)/(2n)^{m+1} < |\lambda|^{n-1}.
\end{aligned} \tag{5.12}$$

By Rouché's Theorem and if we notice (5.10), the proof is completed. $\square$

Let $\widetilde{CP} := \{\widetilde{w}_j = \frac{r_0}{\lambda}\exp(\pi i\frac{2j-1}{m+n}) : 1 \leq j \leq m+n\}$ be the collection of the zeros of $m\lambda^{m+n}z^{m+n}+n=0$, where $r_0 = \sqrt[m+n]{n/m}$. Since $h(x) = x^{1/x}, x > 0$ has maximal value $e^{1/e} < 3/2$ at $x = e$, we have

$$2/3 < 1/\sqrt[m]{m} < r_0 < \sqrt[n]{n} < 3/2. \tag{5.13}$$

The following lemma shows that the rest $m+n$ critical points of P_λ are very "close" to $\widetilde{CP}$.

Lemma 5.3. *If $0 < |\lambda| < 1/(2^m n^2)$, then (5.10) has a solution w_j such that $|w_j - \widetilde{w}_j| < 2(m+n)/m$, where $1 \leq j \leq m+n$. Moreover, $w_i = w_j$ if and only if $i = j$.*

Proof. Dividing $(1+z)^{n-1}$ on both sides of (5.10), we have

$$1+\lambda^{m+n}z^{m+n-1}\left(\frac{m}{n}z+\frac{m+n}{n}\left(1+\frac{n-1}{(1+z)^{n-1}}\right)\right)=0. \tag{5.14}$$

Or, in more useful form

$$\frac{n}{m\lambda^{m+n}}+z^{m+n}+\frac{(m+n)z^{m+n-1}}{m}\left(1+\frac{n-1}{(1+z)^{n-1}}\right)=0. \tag{5.15}$$

Let $\Omega = \{z : |z^{m+n}+\frac{n}{m}\lambda^{-(m+n)}| \leq \beta|\lambda|\cdot\frac{n}{m}|\lambda|^{-(m+n)}\}$, where $\beta = \frac{2(m+n)}{mr_0} < \frac{3(m+n)}{m}$. If $z \in \Omega$, then $|\lambda^{m+n}z^{m+n}+\frac{n}{m}| < \beta|\lambda|\cdot\frac{n}{m}$ and $|z-\widetilde{w}_j| < \beta r_0$ for some $1 \leq j \leq 2n$ by Lemma 2.1(2). If $z \in \Omega$ and $0 < |\lambda| < 1/(2^m n^2)$, we have

$$\frac{n-1}{|1+z|^{n-1}} < \frac{n-1}{((|\lambda|^{-1}-\beta)r_0-1)^{n-1}} < \frac{n-1}{(2^{m+1}n^2/3-3-2n/m)^{n-1}} < \frac{1}{15} \tag{5.16}$$

and

$$\beta|\lambda| \leq \frac{2(m+n)}{2^m n^2\cdot mr_0} < \frac{3}{2^m n}\left(\frac{1}{m}+\frac{1}{n}\right) < \frac{1}{4}, \text{ therefore } \frac{1+\beta|\lambda|}{2(1-\beta|\lambda|)} < \frac{5}{6}. \tag{5.17}$$

Therefore, if $z \in \Omega$ and $0 < |\lambda| < 1/(2^m n^2)$, from (5.16) and (5.17), we have

$$\begin{aligned}
&\left|\frac{(m+n)z^{m+n-1}}{m}\left(1+\frac{n-1}{(1+z)^{n-1}}\right)\right| = \frac{m+n}{m|\lambda|^{m+n}}\left|\frac{\lambda^{m+n}z^{m+n}}{z}\left(1+\frac{n-1}{(1+z)^{n-1}}\right)\right| \\
<\ & \frac{m+n}{m|\lambda|^{m+n}}\,\frac{(\beta|\lambda|+1)n/m}{r_0(1/|\lambda|-\beta)}\cdot\frac{16}{15} = \frac{n\beta|\lambda|}{m|\lambda|^{m+n}}\,\frac{1+\beta|\lambda|}{2(1-\beta|\lambda|)}\cdot\frac{16}{15} < \frac{n\beta|\lambda|}{m|\lambda|^{m+n}}.
\end{aligned} \tag{5.18}$$

Apply Rouché's Theorem to (5.15) and then use Lemma 2.1(2), the proof of the first assertion is completed. By means of the same argument as (2.15), if $0 < |\lambda| < 1/(2^m n^2)$, we have

$$\frac{(r_0/|\lambda|)\cdot\sin(\pi/(m+n))}{2(m+n)/m} \geq \frac{mr_0}{(m+n)^2|\lambda|} > \frac{2^{m+1}m}{3(m/n+1)^2} > 1. \tag{5.19}$$

This means that $w_i = w_j$ if and only if $i = j$. The proof is completed. $\square$

Let $CP := \{w_j : 1 \leq j \leq m+n\}$ be the $m+n$ critical points of P_λ lying near the circle $\mathbb{T}_{r_0/|\lambda|}$ and $CV := \{P_\lambda(w_j) : 1 \leq j \leq m+n\}$. Let CP_{-1} be the collection of $n-1$ critical points of P_λ near -1 (See Lemma 5.2) and $CV_{-1} = \{P_\lambda(z) : z \in CP_{-1}\}$.

Let T_0 be the Fatou component of P_λ containing the attracting petal at the origin and $U := \mathbb{D}(-\frac{3}{4}, \frac{3}{4})$. By Lemmas 5.1(2) and 5.2, we know that $CP_{-1} \cup CV_{-1} \subset U \subset T_0$. Since $P_\lambda(\infty) = -1$, it follows that there exists a neighborhood of ∞ such that P_λ maps it to a neighborhood of -1.

Let T_∞ be the Fatou component such that $\infty \in T_\infty$ and U_0, U_∞ be the component of $P_\lambda^{-1}(U)$ such that $0 \in \overline{U}_0$ and $\infty \in U_\infty$. Obviously, we have $U \subset U_0 \subset T_0$ and $U_\infty \subset T_\infty$.

Lemma 5.4. *If $0 < |\lambda| \leq 1/(2^{10m}n^3)$, there exists an annular neighborhood A_1 of CP containing $\mathbb{T}_{1/|\lambda|} \cup CP$ such that $P_\lambda(A_1) \subset \overline{U'}_\infty \subset U_\infty$, where U'_∞ is a neighborhood of ∞.*

Proof. It is known from Lemma 5.3 that CP is "almost" lying uniformly on the circle $\mathbb{T}_{r_0/|\lambda|}$ and all the finite poles of P_λ lie on the circle $\mathbb{T}_{1/|\lambda|}$. Define annulus

$$A_1 = \{z : 1/(2|\lambda|) < |z| < 2/|\lambda|\}. \tag{5.20}$$

Note that

$$\frac{r_0}{|\lambda|} + \frac{2(m+n)}{m} < \frac{3}{2|\lambda|} + 2 + \frac{2n}{m} < \frac{2}{|\lambda|} \tag{5.21}$$

and

$$\frac{r_0}{|\lambda|} - \frac{2(m+n)}{m} > \frac{2}{3|\lambda|} - 2 - \frac{2n}{m} > \frac{1}{2|\lambda|}. \tag{5.22}$$

We have $\mathbb{T}_{1/|\lambda|} \cup CP \subset A_1$ by Lemma 5.3. If $z \in A_1$ and $|\lambda| \leq \frac{1}{2^{10m}n^3}$, then

$$|P_\lambda(z)+1| \geq \frac{(|z|-1)^n}{n(|\lambda z|^{m+n}+1)} \geq \frac{(\frac{1}{2|\lambda|}-1)^n}{n(2^{m+n}+1)} = \frac{(1-2|\lambda|)^n}{2^n n|\lambda|^n(2^{m+n}+1)} > \frac{2}{|\lambda|^{1+\frac{n}{m}}} + 1. \tag{5.23}$$

In fact,

$$\frac{(1-2|\lambda|)^n}{2^{m+n}+1} > \frac{(1-\frac{2}{2^{10m}n^3})^n}{2^{m+n}+1} > \frac{0.9}{2^{m+n}+1} > \frac{1}{2^{m+n+1}} + 2^n n|\lambda|^n. \tag{5.24}$$

This means that (5.23) follows by

$$2^{m+2n+2}\, n\, |\lambda|^n \leq |\lambda|^{1+n/m}. \tag{5.25}$$

This is true because $|\lambda| \leq \frac{1}{2^{10m}n^3}$. Now we have proved that if $z \in A_1$ and $|\lambda| \leq \frac{1}{2^{10m}n^3}$, then $|P_\lambda(z)| > \frac{2}{|\lambda|^{1+n/m}}$.

On the other hand, if $|z| \geq \frac{2}{|\lambda|^{1+n/m}}$, then

$$|P_\lambda(z)+1| \leq \frac{(|z|+1)^n+1}{|\lambda z|^{m+n}-1} \leq \frac{(1+|z|^{-1})^n+|z|^{-n}}{2^m-|z|^{-n}} < \frac{1}{2}. \tag{5.26}$$

This means that $P_\lambda(z) \in \mathbb{D}(-1,\frac{1}{2}) \subset U$. Let U'_∞ be the component of $P_\lambda^{-1}(\mathbb{D}(-1,\frac{1}{2}))$ containing $\{z : |z| \geq \frac{2}{|\lambda|^{1+n/m}}\}$, it follows that $P_\lambda(A_1) \subset \overline{U'}_\infty \subset U_\infty$ (See Figure 4). $\square$

Proof of Theorem 1.6. For every λ such that $0 < |\lambda| \leq 1/(2^{10m}n^3)$. Let $A := \overline{\mathbb{C}} \setminus (U \cup U'_\infty)$. Since $P_\lambda : U'_\infty \to \mathbb{D}(-1,\frac{1}{2})$ is proper with degree m, it follows that U'_∞ is simply connected and A is an annulus. Note that $P_\lambda^{-1}(U'_\infty)$ is an annulus since there are $m+n$ critical points in $P_\lambda^{-1}(U'_\infty)$ and on which the degree of P_λ is $m+n$. This means that $P_\lambda^{-1}(A)$ consists of two disjoint annuli I_1 and I_2 and $I_1 \cup I_2 \subset A$. The degree of the restriction of P_λ on I_1 and I_2 are m and n respectively.

The following argument is very similar to that of Theorem 1.1. The Julia set of P_λ is $J_\lambda = \bigcap_{k\geq 0} P_\lambda^{-k}(A)$. By the construction, the components of J_n are compact sets nested between -1 and ∞ since $P_\lambda^{-1} : A \to I_j$ is conformal for $j = 1$ or 2. Since the component of J_n can not be a point and the proof of Theorem 1.2 in [PT] can be also applied to geometrically finite rational maps (See [PT, §9] and [TY]), we know that every component of J_n is a Jordan curve. The dynamics of P_λ on the set of Julia components is isomorphic to the one-sided shift on 2 symbols $\Sigma_2 := \{0,1\}^{\mathbb{N}}$. In particular, J_λ is homeomorphic to $\Sigma_2 \times \mathbb{S}^1$, which is a Cantor set of circles as claimed. $\square$

Remark: From the proof of Theorem 1.6 and combine Theorem 3.2, we know that the dynamics on the Julia set of P_λ is conjugated to that of some g_η with the form (1.1). Therefore, we can view P_λ as a "parabolic" McMullen map since the only difference is the sup-attracting basin

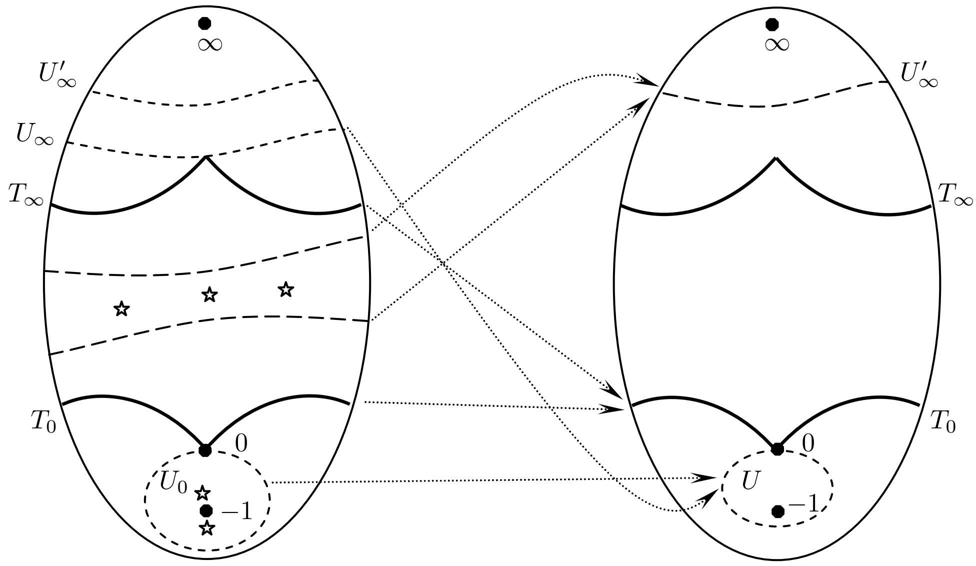

FIGURE 4. Sketch illustrating of the mapping relation of P_λ. The small pentagons denote the critical points.

and its preimages of g_η have been replaced by a fixed parabolic basin and its preimages of P_λ (See Figure 5).

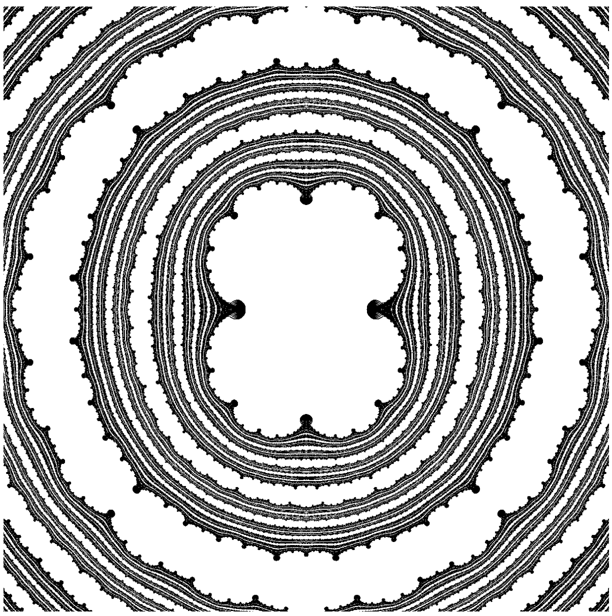

FIGURE 5. The Julia set of P_λ, where $m = 3, n = 2$ and λ is small enough such that J_λ is a Cantor set of circles. All the Fatou components of P_λ are iterated onto the fixed parabolic component (the "cauliflower" in the center of this figure) with parabolic fixed point 1.

6. MORE NON-HYPERBOLIC EXAMPLES

In this section, we will construct more non-hyperbolic rational maps such the Julia sets of them are Cantor circles but they are not included by the previous section. Inspired by Theorem 1.1,

for every $n \geq 2$, we define

$$P_n(z) = A_n \frac{(n+1)z^{(-1)^{n+1}(n+1)}}{nz^{n+1}+1} \prod_{i=1}^{n-1} (z^{2n+2} - b_i^{2n+2})^{(-1)^{i-1}} + B_n, \tag{6.1}$$

where $|b_i| = s^i$ for some $0 < s \leq 1/(25n^2)$ and

$$A_n = \frac{1}{1+(2n+2)C_n} \prod_{i=1}^{n-1} (1 - b_i^{2n+2})^{(-1)^i}, \ B_n = \frac{(2n+2)C_n}{1+(2n+2)C_n} \text{ and } C_n = \sum_{i=1}^{n-1} \frac{(-1)^{i-1} b_i^{2n+2}}{1 - b_i^{2n+2}}. \tag{6.2}$$

Lemma 6.1. (1) $P_n(1) = 1$ *and* $P_n'(1) = 1$.
(2) $1 - s^{2n+1}/(n+1) < |A_n| < 1 + s^{2n+1}/(n+1)$ *and* $|B_n| < s^{2n+1}/(3n+3)$.

Proof. It is easy to see $P_n(1) = 1$ by a straightforward calculation. Note that

$$F_n(z) := \frac{zP_n'(z)}{P_n(z) - B_n} = \sum_{i=1}^{n-1} \frac{(-1)^{i-1}(2n+2)z^{2n+2}}{z^{2n+2} - b_i^{2n+2}} + (-1)^{n+1}(n+1) - \frac{n(n+1)z^{n+1}}{nz^{n+1}+1}. \tag{6.3}$$

This means that

$$\begin{aligned} \frac{P_n'(1)}{P_n(1) - B_n} &= (2n+2) \sum_{i=1}^{n-1} \frac{(-1)^{i-1} b_i^{2n+2}}{1 - b_i^{2n+2}} + (2n+2) \sum_{i=1}^{n-1} (-1)^{i-1} + (-1)^{n+1}(n+1) - n \\ &= (2n+2) \sum_{i=1}^{n-1} \frac{(-1)^{i-1} b_i^{2n+2}}{1 - b_i^{2n+2}} + 1 := (2n+2)C_n + 1. \end{aligned} \tag{6.4}$$

Therefore, we have

$$P_n'(1) = (1 - B_n)((2n+2)C_n + 1) = 1. \tag{6.5}$$

It follows that 1 is a parabolic fixed point of P_n. This completes the proof of (1).

For (2), since $|1 - b_i^{2n+2}|^{-1} \leq 1 + 2|b_1|^{2n+2}$ for $1 \leq i \leq n-1$ and $0 < s \leq 1/(25n^2) \leq 1/100$, then

$$(2n+2)|C_n| < (2n+2)\,(1+2|b_1|^{2n+2}) \sum_{i=1}^{n-1} |b_i|^{2n+2} \leq \frac{(2n+2)(1+2s^{2n+2})s^{2n+2}}{1 - s^{2n+2}} < \frac{s^{2n+1}}{4n+4}. \tag{6.6}$$

We have

$$|B_n| = \left| \frac{(2n+2)C_n}{1+(2n+2)C_n} \right| < (2n+2)|C_n|(1 + (4n+4)|C_n|) < \frac{s^{2n+1}}{3n+3} \tag{6.7}$$

and

$$|A_n| < (1 + (4n+4)|C_n|) \prod_{i=1}^{n-1} (1 + 2|b_i|^{2n+2}) < (1 + \frac{s^{2n+1}}{2n+2})(1 + 5s^{2n+2}) < 1 + \frac{s^{2n+1}}{n+1}. \tag{6.8}$$

Moreover, we have

$$|A_n| > (1 - (2n+2)|C_n|) \prod_{i=1}^{n-1} (1 - |b_i|^{2n+2}) > (1 - \frac{s^{2n+1}}{4n+4})(1 - \frac{s^{2n+2}}{1 - s^{2n+2}}) > 1 - \frac{s^{2n+1}}{n+1}. \tag{6.9}$$

The proof is completed. □

Let us first explain some ideas behind the construction. For $n \geq 2$, define $\widetilde{Q}(z) = (z^{n+1} + n)/(n+1)$ and $\varphi(z) = 1/z$, then $Q(z) := \varphi \circ \widetilde{Q} \circ \varphi^{-1}(z) = (n+1)z^{n+1}/(nz^{n+1}+1)$ satisfies: ∞ is a critical point of Q with multiplicity n which is attracted to the parabolic fixed point 1. Since $\{b_i\}_{1 \leq i \leq n-1}$ are very small, the rational map P_n can be viewed as a small perturbation of Q. The terms A_n and B_n here guarantee that 1 is always a parabolic fixed point of P_n (See Lemma 6.1). It can be shown that P_n maps an annular neighborhood of $\mathbb{T}_{|b_i|}$ into T_0 or T_∞ according to whether i is odd or even, where T_0 and T_∞ denote the Fatou components containing

0 and ∞ respectively (See Lemma 6.5). The Fatou component T_∞ is always parabolic while T_0 is attracting or iterated to T_∞ according to whether n is odd or even. The proof of Theorem 1.7 will based on the mixed arguments as previous 2 sections.

If $|z| \leq 1$, then $|\widetilde{Q}(z)| \leq 1$. This means that the fixed parabolic Fatou component of $\widetilde{Q}$ contains the unit disk for every $n \geq 2$. Therefore, the parabolic Fatou component of Q contains the exterior of the closed unit disk $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Although the polynomial Q has been disturbed into P_n, we still have following

Lemma 6.2. $P_n(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}) \subset (\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}) \cup \{1\}$. *In particular, the disk* $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ *lies in the parabolic Fatou component of* P_n *with parabolic fixed point* 1.

The proof of Lemma 6.2 is very subtle, which will be delayed to next section.

Lemma 6.3. *For* $n \geq 2$ *and* $1 \leq i \leq n-1$, *then*

$$\sum_{1 \leq j < i} (-1)^j + \sum_{i < j \leq n-1} (-1)^{j-1} + \frac{1+(-1)^{n+1}}{2} = 0. \tag{6.10}$$

Proof. The argument is based on several cases showed in Table 1. $\square$

		$\sum_{1\leqslant j<i}(-1)^j$	$\sum_{i<j\leqslant n-1}(-1)^{j-1}$	$(1+(-1)^{n+1})/2$
odd n	odd i	0	-1	1
	even i	-1	0	1
even n	odd i	0	0	0
	even i	-1	1	0

Table 1.

As before, we first locate the critical points of P_n. Note that 0 and ∞ are both critical points of P_n with multiplicity n and the degree of P_n is n^2+n. The rest $2(n^2-1)$ critical points of P_n are the solutions of $F_n(z)=0$ (See (6.3)).

For $1 \leq i \leq n-1$, let $\widetilde{CP}_i := \{\widetilde{w}_{i,j} = b_i \exp(\pi i \frac{2j-1}{2n+2}) : 1 \leq j \leq 2n+2\}$ be the collection of $2n+2$ points lying on $\mathbb{T}_{|b_i|}$ uniformly. The following lemma is similar to Lemmas 2.3 and 5.3.

Lemma 6.4. *For every* $\widetilde{w}_{i,j} \in \widetilde{CP}_i$, *where* $1 \leq i \leq n-1$ *and* $1 \leq j \leq 2n+2$, *there exists* $w_{i,j}$, *which is a solution of* $F_n(z)=0$, *such that* $|w_{i,j} - \widetilde{w}_{i,j}| < s^{n+1/2}|b_i|$. *Moreover,* $w_{i_1,j_1} = w_{i_2,j_2}$ *if and only if* $(i_1,j_1)=(i_2,j_2)$.

Proof. Note that $F_n(z)=0$ is equivalent to

$$\sum_{i=1}^{n-1} (-1)^{i-1} \frac{z^{2n+2}+b_i^{2n+2}}{z^{2n+2}-b_i^{2n+2}} + \frac{1+(-1)^{n+1}}{2} - \frac{nz^{n+1}}{nz^{n+1}+1} = 0. \tag{6.11}$$

Timing $z^{2n+2}-b_i^{2n+2}$ on both sides of (6.11), where $1 \leq i \leq n-1$, we have

$$(-1)^{i-1}(z^{2n+2}+b_i^{2n+2}) + (z^{2n+2}-b_i^{2n+2})\, G_i(z) = 0, \tag{6.12}$$

where

$$G_i(z) = \sum_{1 \leq j \leq n-1,\, j \neq i} (-1)^{j-1} \frac{z^{2n+2}+b_j^{2n+2}}{z^{2n+2}-b_j^{2n+2}} + \frac{1+(-1)^{n+1}}{2} - \frac{nz^{n+1}}{nz^{n+1}+1}. \tag{6.13}$$

Let $\Omega_i = \{z : |z^{2n+2}+b_i^{2n+2}| \leq s^{n+1/2}|b_i|^{2n+2}\}$, where $1 \leq i \leq n-1$. If $z \in \Omega_i$, then $|z|^{n+1} \leq (1+s^{n+1/2})|b_i|^{n+1} \leq (1+s^{n+1/2})s^{n+1}$ by Lemma 2.1(2). So

$$\left|\frac{nz^{n+1}}{nz^{n+1}+1}\right| \leq \frac{n(1+s^{n+1/2})s^{n+1}}{1-n(1+s^{n+1/2})s^{n+1}} \leq \frac{(1+100^{-5/2})s^{n+1/2}/5}{1-(1+100^{-5/2})100^{-5/2}/5} < 0.3\, s^{n+1/2}$$

since $s \le 1/(25n^2) \le 1/100$. For every $z \in \Omega_i$, if $1 \le j < i$, we have

$$|z/b_j|^{2n+2} = |z/b_i|^{2n+2}|b_i/b_j|^{2n+2} < (1+s^{n+1/2})\, s^{(2n+2)(i-j)}. \tag{6.14}$$

If $i < j \le n-1$, by the first statement of Lemma 2.1(2), we have

$$|b_j/z|^{2n+2} = |b_i/z|^{2n+2}|b_j/b_i|^{2n+2} \le (1+2\cdot s^{n+1/2})\, s^{(2n+2)(j-i)}. \tag{6.15}$$

From (6.14), (6.15) and Lemma 6.3, we have

$$\begin{aligned} &\left|G_i(z) + \frac{nz^{n+1}}{nz^{n+1}+1}\right| \\ =& \left|\sum_{1\le j<i}(-1)^j\frac{1+(z/b_j)^{2n+2}}{1-(z/b_j)^{2n+2}} + \sum_{i<j\le n-1}(-1)^{j-1}\frac{1+(b_j/z)^{2n+2}}{1-(b_j/z)^{2n+2}} + \frac{1+(-1)^{n+1}}{2}\right| \\ <& \ 3\cdot(1+2\cdot s^{n+1/2})\left(\sum_{1\le j<i} s^{(2n+2)(i-j)} + \sum_{i<j\le n-1} s^{(2n+2)(j-i)}\right) \\ <& \ 6\cdot(1+2\cdot s^{n+1/2})^2\, s^{2n+2}. \end{aligned} \tag{6.16}$$

The first inequality in (6.16) is benefit from the inequality $2x/(1-x) \le 3x$ if $x < 1/3$ (Here $x \le (1+2\cdot s^{n+1/2})\, s^{2n+2} < 10^{-10}$). So we have

$$|G_i(z)| < \ 6\cdot(1+2\cdot s^{n+1/2})^2\, s^{2n+2} + 0.3\, s^{n+1/2} < 0.4\, s^{n+1/2}. \tag{6.17}$$

Therefore, if $z \in \Omega_i$, then

$$|z^{2n+2} - b_i^{2n+2}|\cdot|\,G_i(z)| < (2+s^{n+1/2})|b_i|^{2n+2}\cdot 0.4\, s^{n+1/2} < s^{n+1/2}|b_i|^{2n+2}. \tag{6.18}$$

From (6.12) and by Rouché's Theorem, there exists a solution $w_{i,j}$ of $F_n(z) = 0$ such that $w_{i,j} \in \Omega_i$ for every $1 \le j \le 2n+2$. In particular, $|w_{i,j} - \widetilde{w}_{i,j}| < s^{n+1/2}|b_i|$ by the second statement of Lemma 2.1(2). The assertion $w_{i_1,j_1} = w_{i_2,j_2}$ if and only if $(i_1,j_1) = (i_2,j_2)$ can be verified similarly as (2.14) and (2.15). The proof is completed. $\square$

For $1 \le i \le n-1$, let $CP_i := \{w_{i,j} : 1 \le j \le 2n+2\}$ be the collection of critical points of P_n which lies closely to the circle $\mathbb{T}_{|b_i|}$.

Lemma 6.5. *There exist $n-1$ annuli $\{A_i\}_{i=1}^{n-1}$ satisfying $A_{n-1} \prec \cdots \prec A_1$ and two simply connected domain U_0 and U_∞ which contains 0 and ∞ respectively, such that*

(1) *$U_\infty \supset \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and $P_n(\overline{U}_\infty) \subset U_\infty \cup \{1\}$;*
(2) *$A_i \supset \mathbb{T}_{|b_i|} \cup CP_i$, $P_n(\overline{A}_i) \subset U_0$ for odd i and $P_n(\overline{A}_i) \subset U_\infty$ for even i;*
(3) *$P_n(\overline{U}_0) \subset U_\infty$ for even n and $P_n(\overline{U}_0) \subset U_0$ for odd n.*

Proof. Let $U_\infty := \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ be the exterior of the closed unit disk. Then (1) is obvious if we notice Lemma 6.2. Let $\varepsilon = s^{n+1/2}$ and $A_i = \mathbb{A}_{|b_i|(1-2\varepsilon),|b_i|(1+2\varepsilon)}$. From (6.1), we know that

$$|R_n(z)| := \left|\frac{P_n(z)-B_n}{A_n}\cdot\frac{nz^{n+1}+1}{n+1}\right| = |z|^{(-1)^{n+1}(n+1)}\,|z^{2n+2}-b_i^{2n+2}|^{(-1)^{i-1}}H_i(z), \tag{6.19}$$

where

$$H_i(z) = \prod_{j=1}^{i-1}|b_j|^{(2n+2)(-1)^{j-1}}\prod_{j=i+1}^{n-1}|z|^{(2n+2)(-1)^{j-1}}\cdot Q_i(z) \tag{6.20}$$

and

$$Q_i(z) = \prod_{j=1}^{i-1}\left|1-(z/b_j)^{2n+2}\right|^{(-1)^{j-1}}\prod_{j=i+1}^{n-1}\left|1-(b_j/z)^{2n+2}\right|^{(-1)^{j-1}}. \tag{6.21}$$

If $z \in \overline{A}_i$, where $1 \le i \le n-1$, we have

$$Q_i(z) < \prod_{j=1}^{i-1} \left(1 + 3\,|b_i/b_j|^{2n+2}\right) \prod_{j=i+1}^{n-1} \left(1 + 3\,|b_j/b_i|^{2n+2}\right) < (1 + 6s^{2n+2})^2 \tag{6.22}$$

and

$$Q_i(z) > \prod_{j=1}^{i-1} \left(1 + 3\,|b_i/b_j|^{2n+2}\right)^{-1} \prod_{j=i+1}^{n-1} \left(1 + 3\,|b_j/b_i|^{2n+2}\right)^{-1} > (1 + 6s^{2n+2})^{-2}. \tag{6.23}$$

Note that $\varepsilon = s^{n+1/2} \le (5n)^{-2n-1} \le 10^{-5}$. If n is even and $1 \le i \le n-1$ is odd, then for $z \in \overline{A}_i$, we have

$$\begin{aligned}|R_n(z)| = {} & \frac{|z^{2n+2} - b_i^{2n+2}|}{|z|^{n+1}} \frac{1}{s^{(i-1)(n+1)}}\, Q_i(z) < \frac{|b_i|^{n+1}(1 + (1+2\varepsilon)^{2n+2})}{(1-2\varepsilon)^{n+1}} \frac{(1+6s^{2n+2})^2}{s^{(i-1)(n+1)}} \\ = {} & \frac{1 + (1+2\varepsilon)^{2n+2}}{(1-2\varepsilon)^{n+1}} (1 + 6s^{2n+2})^2 s^{n+1} < 2.1 \cdot s^{n+1}.\end{aligned}$$

If n and $1 \le i \le n-1$ are both even, then for $z \in \overline{A}_i$, we have

$$|R_n(z)| = \frac{|b_{i-1}|^{2n+2}|z|^{2n+2}}{|z|^{n+1}|z^{2n+2} - b_i^{2n+2}|} \frac{1}{s^{(i-2)(n+1)}}\, Q_i(z) > \frac{(1-2\varepsilon)^{n+1}}{1 + (1+2\varepsilon)^{2n+2}} (1 - 6s^{2n+2})^2 > 0.49.$$

This means that if n is even and $1 \le i \le n-1$ is odd, for $z \in \overline{A}_i$, we have

$$|P_n(z)| < \left|\frac{2.1 \cdot s^{n+1} \cdot (n+1)\, A_n}{nz^{n+1} + 1}\right| + |B_n| \le \frac{2.1\, (s^{n+1/2}/5) \cdot (1 + s^{2n+1}/(n+1))}{1 - n(1+2\varepsilon)s^{n+1}} + \frac{s^{2n+1}}{3n+3} < s^{n+1/2}$$

by Lemma 6.1(2). If n and $1 \le i \le n-1$ are both even, then for $z \in \overline{A}_i$, we have

$$|P_n(z)| > \left|\frac{0.49(n+1)A_n}{nz^{n+1} + 1}\right| - |B_n| \ge \frac{0.49(n+1)(1 - s^{2n+1}/(n+1))}{1 + n(1+2\varepsilon)s^{n+1}} - \frac{s^{2n+1}}{3n+3} > \frac{n+1}{3} \ge 1.$$

By the completely similar arguments, one can show that if n is odd, for $z \in \overline{A}_i$, we have

$$|P_n(z)| < s^{n+1/2} \text{ for odd } i \text{ and } |P_n(z)| > 1 \text{ for even } i. \tag{6.24}$$

Let $U_0 = \mathbb{D}_r$, where $r = s^{n+1/2}$. This proves (2).

If n is odd, for every z such that $|z| \le s^{n+1/2}$, we have

$$\begin{aligned}|P_n(z)| \le {} & \left|\frac{(n+1)A_n}{nz^{n+1} + 1}\right| \, |z|^{n+1} \prod_{i=1}^{n-1} |b_i|^{(2n+2)(-1)^{i-1}} \prod_{i=1}^{n-1} \left|1 - \frac{z^{2n+2}}{b_i^{2n+2}}\right|^{(-1)^{i-1}} + |B_n| \\ \le {} & \frac{(n+1)(1 + s^{2n+1}/(n+1))}{1 - ns^{n^2+n/2}}\, s^{3(n+1)/2} \prod_{i=1}^{n-1} \left(1 + 2\frac{|z|^{2n+2}}{|b_i|^{2n+2}}\right) + \frac{s^{2n+1}}{3n+3} < s^{n+1/2}.\end{aligned}$$

It follows that $P_n(\overline{\mathbb{D}}_r) \subset \mathbb{D}_r$ for odd n, where $r = s^{n+1/2}$.

If n is even, then P_n maps a neighborhood of 0 to that of ∞. For every z such that $|z| \le s^{n+1/2}$, we have

$$|P_n(z)| \ge \frac{(n+1)\, s^{-(n+1)/2}\, (1 - s^{2n+1}/(n+1))}{1 + ns^{n^2+n/2}} \prod_{i=1}^{n-1} \left(1 - 2\frac{|z|^{2n+2}}{|b_i|^{2n+2}}\right) - \frac{s^{2n+1}}{3n+3} > n > 1. \tag{6.25}$$

This ends the proof of (3). The proof is completed. □

Proof of Theorem 1.7. Let $A := \overline{\mathbb{C}} \setminus (U_0 \cup U_\infty)$. The Julia set of P_n is equal to $\bigcap_{k\ge0} P_n^{-k}(A)$. Note that P_n is geometrically finite. The argument is completely similar to the proofs of Theorems 1.1 and 1.6. The set of Julia components of P_n is isomorphic to the one-sided shift on n symbols $\Sigma_n := \{0, 1, \cdots, n-1\}^{\mathbb{N}}$. In particular, the Julia set of P_n is homeomorphic to $\Sigma_n \times \mathbb{S}^1$, which is a Cantor set of circles as desired (See Figure 6). We omit the details here. □

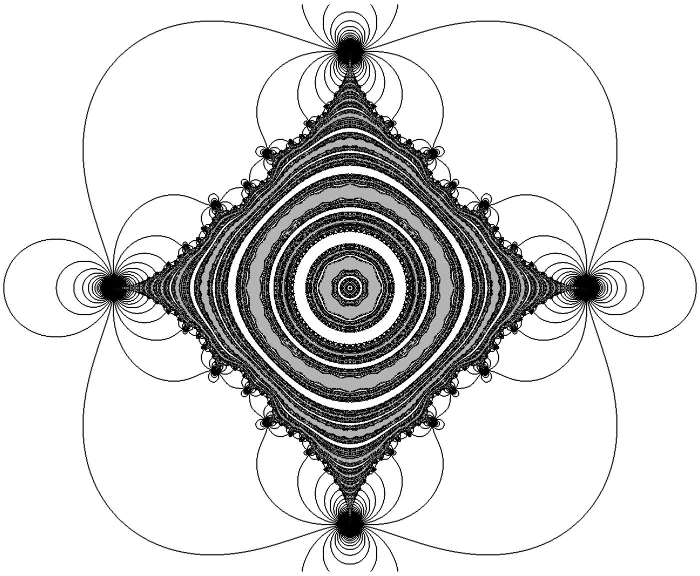

FIGURE 6. The Julia set of P_3, which is a Cantor set of circles. The parameter s is chosen small enough. The gray parts in the Figure denote the Fatou components which are iterated to the attracting Fatou component containing the origin, while the white parts denote the Fatou components iterated to the parabolic Fatou component whose boundary contains the parabolic fixed point 1. Some equipotentials of Fatou coordinate have been drawn in the parabolic Fatou component and its preimages. Figure range: $[-1.6, 1.6] \times [-1.2, 1.2]$.

7. PROOF OF LEMMA 6.2

This section will devote to proving Lemma 6.2, which is the key ingredient in the proof of Lemma 6.5 and hence in Theorem 1.7.

Proof. Let $\widetilde{R}(z) = 1/P_n(1/z)$, then Lemma 6.2 is reduced to proving $\widetilde{R}(\overline{\mathbb{D}}) \subset \mathbb{D} \cup \{1\}$. Let $w = z^{n+1}$, by a straightforward calculation, we have

$$R(w) := \widetilde{R}(z) = \frac{w+n}{n+1} \cdot \frac{1}{S(w)}, \tag{7.1}$$

where

$$S(w) = A_n \prod_{i=1}^{n-1} (1 - b_i^{2n+2} w^2)^{(-1)^{i-1}} + \frac{w+n}{n+1} B_n = 1 + \frac{w-1}{1+(2n+2)C_n} \left(\frac{H(w)-1}{w-1} + 2C_n \right) \tag{7.2}$$

and

$$H(w) = \prod_{i=1}^{n-1} (1 - b_i^{2n+2})^{(-1)^i} \prod_{i=1}^{n-1} (1 - b_i^{2n+2} w^2)^{(-1)^{i-1}}. \tag{7.3}$$

Since $H(1) = 1$, it follows that $H'(1)$ is a finite number. In fact,

$$I(w) := \frac{H'(w)}{H(w)} = -2w \sum_{i=1}^{n-1} \frac{(-1)^{i-1} b_i^{2n+2}}{1 - b_i^{2n+2} w^2}. \tag{7.4}$$

We know that $I(1) = H'(1) = -2C_n$. For every small enough $w - 1$, we can write $S(w)$ as

$$S(w) = 1 + \frac{(w-1)^2}{1+(2n+2)C_n} \cdot \frac{\frac{H(w)-1}{w-1} + 2C_n}{w-1} =: 1 + \frac{(w-1)^2}{1+(2n+2)C_n} \cdot \Phi(w), \tag{7.5}$$

where

$$\Phi(w)=\sum_{k\geq 2}\frac{H^{(k)}(1)}{k!}(w-1)^{k-2}. \tag{7.6}$$

The next step is to estimate $H^{(k)}(1)$ for every $k\geq 2$.

For every $k\geq 1$, let

$$Y_k(w)=\sum_{i=1}^{n-1}(-1)^{i-1}\left(\frac{b_i^{2n+2}}{1-b_i^{2n+2}w^2}\right)^k. \tag{7.7}$$

In particular, $Y_1(1)=C_n$ and

$$Y_k'(w)=2kw\,Y_{k+1}(w). \tag{7.8}$$

If $|w|=1$, we have

$$|Y_k(w)|\leq\left|\frac{b_1^{2n+2}}{1-b_1^{2n+2}}\right|^k\left(1+\sum_{i=2}^{n-1}\left|\frac{b_i^{2n+2}(1-b_1^{2n+2})}{b_1^{2n+2}(1-b_i^{2n+2})}\right|^k\right)\leq\frac{11}{10}\left|\frac{b_1^{2n+2}}{1-b_1^{2n+2}}\right|^k. \tag{7.9}$$

Similarly, we have $|Y_k(w)|\geq\frac{9}{10}|b_1^{2n+2}/(1-b_1^{2n+2})|^k$. This means that

$$\left|\frac{Y_{k+1}(w)}{Y_k(w)}\right|\leq\frac{11}{9}\left|\frac{b_1^{2n+2}}{1-b_1^{2n+2}}\right|\leq 2s^{2n+2}<1/2. \tag{7.10}$$

We first claim that $|I^{(k)}(1)|\leq 2^{k+1}k!|C_n|$ for every $k\geq 0$. Since $I^{(0)}(w)=-2wY_1(w)$ and $I^{(1)}(w)=-2Y_1(w)-4w^2Y_2(w)$, it can be proved inductively that $I^{(k)}(w)$ can be written as

$$I^{(k)}(w)=\sum_{j=1}^{2^k}Q_{k,j}(w)=\sum_{j=1}^{2^k}P_{k,j}(w)Y_{k,j}(w), \tag{7.11}$$

where $P_{k,j}(w)$ is a polynomial with degree at most $k+1$ and $Y_{k,j}=Y_l$ for some $1\leq l\leq k+1$. Note that some terms $Q_{k,j}$ may be equal to zero (the degree of corresponding polynomial $P_{k,j}$ is regarded as $-\infty$) and the formula (7.11) can be simplified, but what we need is this "long" expansion. In particular, without loss of generality, for $1\leq j\leq 2^k$, we require further that

$$P_{k+1,2j-1}(w)Y_{k+1,2j-1}(w)=P_{k,j}'(w)Y_{k,j}(w)\text{ and }P_{k+1,2j}(w)Y_{k+1,2j}(w)=P_{k,j}(w)Y_{k,j}'(w). \tag{7.12}$$

Since $\deg(P_{k,j})\leq k+1$ and $Y_{k,j}=Y_l$ for some $1\leq l\leq k+1$, it follows that

$$\begin{aligned}&|P_{k+1,2j-1}(1)Y_{k+1,2j-1}(1)|+|P_{k+1,2j}(1)Y_{k+1,2j}(1)|\\=\ &|P_{k,j}'(1)Y_l(1)|+|P_{k,j}(1)Y_l'(1)|\\\leq\ &(k+1)|P_{k,j}(1)Y_l(1)|+2(k+1)|P_{k,j}(1)Y_{l+1}(1)|\\\leq\ &2(k+1)|P_{k,j}(1)Y_{k,j}(1)|\end{aligned} \tag{7.13}$$

since $|Y_{l+1}(1)/Y_l(1)|\leq 1/2$ for every $l\geq 1$ by (7.10).

Denote $||I^{(k)}(1)||:=\sum_{j=1}^{2^k}|P_{k,j}(1)Y_{k,j}(1)|$, we have $||I^{(k)}(1)||\leq 2k||I^{(k-1)}(1)||$. This means that

$$|I^{(k)}(1)|\leq||I^{(k)}(1)||\leq 2^kk!||I^{(0)}(1)||=2^{k+1}k!|C_n|. \tag{7.14}$$

This proves the claim $|I^{(k)}(1)|\leq 2^{k+1}k!|C_n|$ for every $k\geq 0$.

Secondly, we check by induction that $|H^{(k)}(1)|\leq 4^kk!|C_n|$ for $k\geq 1$. For $k=1$, we have $|H'(1)|=2|C_n|<4|C_n|$. Assume that $|H^{(i)}(1)|\leq 4^ii!|C_n|$ for every $1\leq i\leq k$. By (7.4), we have $H'(w)=H(w)I(w)$. So

$$\begin{aligned}|H^{(k+1)}(1)|&\leq|I^{(k)}(1)|+\sum_{i=1}^{k}\frac{k!}{i!(k-i)!}|H^{(i)}(1)|\cdot|I^{(k-i)}(1)|\\&\leq 2^{k+1}k!|C_n|(1+2^{k+1}|C_n|)\leq 4^{k+1}(k+1)!|C_n|\end{aligned} \tag{7.15}$$

since $|I^{(k-i)}(1)|\leq 2^{k-i+1}(k-i)!|C_n|$ and $|H^{(i)}(1)|\leq 4^ii!|C_n|$ for every $1\leq i\leq k$.

If $w = e^{i\theta}$ for $|\theta| \le 1/20$, then $|w-1| < |\theta| \le 1/20$. By (7.6) and (7.15), we have

$$|\Phi(w)| \le \sum_{k\ge 2} 4^k |C_n| (1/20)^{k-2} \le 16|C_n| \sum_{k\ge 0} 5^{-k} = 20|C_n|. \tag{7.16}$$

By (7.5) and (7.16), it follows that

$$|S(w)| \ge 1 - \frac{\theta^2}{1-(2n+2)|C_n|} 20|C_n| \ge 1 - \frac{s^{2n+1}}{n+1}\theta^2 \tag{7.17}$$

since $n \ge 2$ and $|C_n| < s^{2n+1}/(8(n+1)^2)$ by (6.6).

On the other hand, if $w = e^{i\theta}$ for $0 \le |\theta| \le \pi$, then

$$\left|\frac{w+n}{n+1}\right| = \left(1 - \frac{4n}{(n+1)^2}\sin^2\frac{\theta}{2}\right)^{1/2} \le \left(1 - \frac{4n}{\pi^2(n+1)^2}\theta^2\right)^{1/2} \le 1 - \frac{2n}{(n+1)^2\pi^2}\theta^2 \tag{7.18}$$

since $(1-x)^{1/2} \le 1 - x/2$ for $0 \le x < 1$. This means that if $w = e^{i\theta}$ for $|\theta| \le 1/20$, then

$$|R(w)| \le (1 - \frac{2n}{(n+1)^2\pi^2}\theta^2)(1 - \frac{s^{2n+1}}{n+1}\theta^2)^{-1} \le 1. \tag{7.19}$$

Moreover, $|R(w)| = 1$ if and only if $w = 1$.

If $w = e^{i\theta}$ for $|\theta| > 1/20$, by (7.2) and Lemma 6.1(2), we have

$$|S(w)| \ge (1 - \frac{s^{2n+1}}{n+1}) \prod_{i=1}^{n-1} (1 - |b_i|^{2n+2}) - \frac{s^{2n+1}}{3n+3} \ge 1 - \frac{3s^{2n+1}}{n+1}. \tag{7.20}$$

By (7.18) and (7.20), we have

$$|R(w)| \le (1 - \frac{2}{20^2(n+1)\pi^2})(1 - \frac{3s^{2n+1}}{n+1})^{-1} < 1. \tag{7.21}$$

This means that $R(w)$ maps the boundary of the unit disk into the unit disk except at $w = 1$. Since $R(w) \ne \infty$ if $|w| \le 1$, we know that $R(\overline{\mathbb{D}}) \subset \mathbb{D} \cup \{1\}$. Therefore, $\widetilde{R}(\overline{\mathbb{D}}) \subset \mathbb{D} \cup \{1\}$ and $\widetilde{R}$ maps $\{z \in \mathbb{C} : z^{n+1} = 1\}$ onto 1. This ends the proof of Lemma 6.2. □

References

[Be] A. Beardon, *Iteration of rational functions*. Grad. Texts Math., **132**, Springer-Verlag, 1991.

[Cui] G. Cui, Dynamics of rational maps, topology, deformation and bifurcation, preprint, May 2002 (Early version: Geometrically finite rational maps with given combinatorics, 1997).

[DLU] R. Devaney, D. Look and D. Uminsky, The Escape Trichotomy for Singularly Perturbed Rational Maps, Indiana University Mathematics Journal **54** (2005), 1621–1634.

[HP] P. Haïssinsky and K. Pilgrim, Quasisymmetrically inequivalent hyperbolic Julia sets. To appear in Revista Math. Iberoamericana, 2012.

[MT] J. Mackay and J. Tyson, *Conformal dimension: Theory and applications*, University Lecture Series, 54. American Mathematical Society, Providence, RI, 2010.

[Mc] C. McMullen, Automorphisms of rational maps. in *Holomorphic Functions and Moduli I*, Math. Sci. Res. Inst. Publ. **10**, Springer, 1988.

[PT] K. Pilgrim and L. Tan, Rational maps with disconnected Julia sets. Asterisque, **261** (2000), 349–383.

[QWY] W. Qiu, X. Wang and Y. Yin, Dynamics of McMullen maps. Advances in Mathematics, **229** (2012), 2525–2577.

[QY] W. Qiu and Y. Yin, Proof of the Branner-Hubbard conjecture on Cantor Julia sets. Science in China, Series A, **52** (2009), No. 1, 45–65.

[St] N. Steinmatz, On the dynamics of McMullen family. Conformal Geometry and Dynamics, **10** (2006) 159–183.

[Su] D. Sullivan, Conformal dynamical systems. In *Geometric Dynamics*. Leture Notes No. 1007, Springer Verlag, 1983.

[TY] L. Tan and Y. Yin, Local connectivity of the Julia sets for geometrically finite rational maps. Science in China, Series A (1) **39** (1996), 39–47.

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